Wallis sequence estimated through the Euler–Maclaurin formula: even from the Wallis product π could be computed fairly accurately

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Summary

The power of the Euler–Maclaurin summation formula is illustrated through the example in which π is computed quite accurately from the slowly convergent Wallis sequence $W(n) := \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}$. Using the Euler–Maclaurin formula the rate of convergence of $W(n)$ is estimated as $\frac{3}{10n} < \frac{\pi}{2} - W(n) < \frac{4}{10n}$, for integer $n \geq 3$. Presented example implicitly suggests that perhaps in the undergraduate mathematics curriculum the Euler–Maclaurin formula of a lower order should be included, or simply replacing the Simpson's formula.

Key words: acceleration of convergence, Euler–Maclaurin summation, the rate of convergence, Wallis product.
MSC: 40A05, 40A20, 40A25, 65B10, 65B15.

1 Introduction

The first presentation of $\pi := \frac{\text{circumference of a circle}}{\text{diameter of a circle}}$ in a form of a limit has been made by Wallis[1] in 1655, see [22] and [24]. This presentation can be obtained in the following way.

For the sequence

$$I_k := \int_0^{\frac{\pi}{2}} \sin^k x \, dx$$

we have $I_0 = \frac{\pi}{2}$, $I_1 = 1$, and, using the method of integration by parts,

$$I_k = \frac{k-1}{k} I_{k-2} \quad \text{for } k \geq 2.$$

Consequently, putting $k = 2n$ and $k = 2n+1$ $(n \in \mathbb{N})$, we find, by induction, the next two expressions

$$I_{2n} = \prod_{k=1}^{n} \frac{2k-1}{2k} \cdot \frac{\pi}{2}$$

and

$$I_{2n+1} = \prod_{k=1}^{n} \frac{2k}{2k+1} \, .$$

Because

$$0 < \sin^{2n+2} x < \sin^{2n+1} x < \sin^{2n} x < 1 \quad \text{for } x \in \left(0, \frac{\pi}{2}\right) \text{ and } n \in \mathbb{N},$$

[1]John Wallis, 1616 – 1703, English mathematician

we have

$$0 \;<\; I_{2n+2} \;<\; I_{2n+1} \;<\; I_{2n} \;<\; 1$$

for all $n \in \mathbb{N}$. Hence, due to the previous expressions for I_{2n} and I_{2n+1}, the estimate

$$\prod_{k=1}^{n+1} \frac{2k-1}{2k} \cdot \frac{\pi}{2} \;<\; \prod_{k=1}^{n} \frac{2k}{2k+1} \;<\; \prod_{k=1}^{n} \frac{2k-1}{2k} \cdot \frac{\pi}{2}$$

holds. Therefore

$$\prod_{k=1}^{n} \frac{2k}{2k-1} \cdot \prod_{k=1}^{n} \frac{2k}{2k+1} \;<\; \frac{\pi}{2} \;<\; \prod_{k=1}^{n+1} \frac{2k}{2k-1} \cdot \prod_{k=1}^{n} \frac{2k}{2k+1}.$$

or

$$\left(\prod_{k=1}^{n} \frac{4k^2}{4k^2-1}\right) \;<\; \frac{\pi}{2} \;<\; \left(\prod_{k=1}^{n} \frac{4k^2}{4k^2-1}\right) \cdot \frac{2n+2}{2n+1}$$

for every $n \in \mathbb{N}$. Thus, the sequence

$$n \mapsto W(n) := \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \equiv \frac{1}{2n+1} \left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right)^2, \tag{1}$$

involving only rational numbers, converges towards the infinite Wallis product

$$W(\infty) := \lim_{n\to\infty} W(n) = \prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} = 2 \prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k+1)^2}\right) = \frac{\pi}{2}\,; \tag{2a}$$

see [1, p. 258], [2, p. 384],[7, p. 213], [14, pp. 5&47], [16, p. 384], [19, pp. 14&465] or [22]. This old and interesting formula seems to be of small practical value for numerical computation of π as it is frequently noted in the calculus textbooks. In fact, the Wallis sequence converges very slowly and its late terms are directly not easily computable. However, even from the Wallis product, π could be computed fairly accurately. Indeed, expressing the Wallis product by the Gamma function [1, pp. 255&258] as

$$W(n) = \frac{\pi}{2n+1} \frac{\Gamma^2(n+1)}{\Gamma^2(n+\frac{1}{2})} = \frac{\pi n^2}{2n+1} \left(\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}\right)^2$$

and then using the continuous version of Stirling's formula[2] [1, p. 257]

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x \frac{\Theta_x}{12x}$$

for $x > 0$ and some $\Theta_x \in (0,1)$, we obtain the expression

$$W(n) = \frac{1}{2}\,\pi e \left(1 - \frac{1}{2n+1}\right)^{2n+1} \exp\left(\frac{\Theta_n}{6n} - \frac{\Theta'_n}{6n+3}\right) \tag{2b}$$

and consequently also

$$\pi = W(n) \left(2 + \frac{1}{n}\right) e^{-1} \left(1 + \frac{1}{2n}\right)^{2n} \exp\left(\frac{\Theta'_n}{6n+3} - \frac{\Theta_n}{6n}\right) \tag{2c}$$

for every positive integer n and some $\Theta_n, \Theta'_n \in (0,1)$. From this last formula it is possible to compute π to several decimal places and from preceding formula we can compute Wallis products also for large values of n.

[2]The constant C figured in the original version of Stirling's formula, $\Gamma(x) = \frac{C}{\sqrt{x}} \left(\frac{x}{e}\right)^x \frac{\Theta_x}{12x}$, was in fact determined just using the Wallis product as $C = \sqrt{2\pi}$. This expression for C was still unknown to Stirling.

In the lines above we have used two important theorems concerning Gamma function, the last one was the Stirling's formula, which plays the crucial role in the approximations given above. However, this theorem can be deduced using special technique of summation known as the Euler–Maclaurin summation formula, see for example [1, 12, 13, 14, 16, 17, 25]. Although this formula is very important, it is not included, contrary to our opinion, in the undergraduate curriculum. Perhaps due to the fact that many a mathematician believes that the Euler–Maclaurin formula, as well as its derivation, is too complicated to enter into the undergraduate curriculum. Is it true? Certainly not, as was clearly shown in [17] and [18]. Moreover, using only the Euler–Maclaurin formula of order 3, the Wallis sequence, as well as π, could be estimated directly rather well, similarly to (2b) and (2c), avoiding all the complications quoted above. Hence, we wish to present the power of the Euler–Maclaurin tool and simultaneously complement the article [17] with this example. Using this formula we can often successfully estimate definite integrals, sums and products as well. In the present article we use also the Taylor's formula, which is also deduced in [17].

We wish to point out to teachers what powerful device they are overlooking. We plead for the Euler–Maclaurin formula of order $p \leq 4$ to be included into the universities/colleges undergraduate curriculum. The absence of this formula in the mathematics curriculum is similar to the absence of DNA technology in forensic science. Namely, as it is possible to deduce from only one hair, found on the victim, a lot of crucial conclusions, similarly we can in some cases compute the sum of a slowly convergent series using the Euler–Maclaurin formula, and even much more, see [17]. In the present article we illustrate this fact, where we transform the slowly convergent Wallis sequence into the faster convergent sequence with limit equal to $\frac{\pi}{2}$. This way we introduce the numerical summation technique to a wide community of readers. The presented technique should attract teacher/student into a sphere of constructive (concrete) mathematics. Derivation of the Euler–Maclaurin formula of low order, for example of order $p \leq 4$, can be carried out quite easily, even more easily than it was done in [17]; see also [18]. But with such a formula we can compare integral and its integral sum, which is interesting for a teacher and for a student as well. For example, from (6), (7a), (7b) and (10) below we extract the Euler–Maclaurin formula of order $p = 3$:

$$\sum_{k=m}^{n-1} f(k) - \int_m^n f(x)\,dx = -\frac{1}{2}\left[f(n) - f(m)\right] + \frac{1}{12}\left[f'(n) - f'(m)\right] - \frac{1}{6}\int_m^n P_3\left(-x\right)\, f^{(3)}(x)\,dx,$$

where P_3 is 1-periodic, differentiable function, bounded as $|P_3(x)| < \frac{1}{20}$ for $x \in \mathbb{R}$. So, if $\left|f^{(3)}(x)\right|$ is small, we can make useful estimate of the difference between the integral and its integral sum. This way we produce formulas for numerical integration and summation simultaneously. It is interesting that the Euler–Maclaurin formula of order 4 gives for the absolute value of the remainder in the numerical integration rule (Hermite's rule) four times smaller a priori estimate as it is known for the Simpson's rule. Moreover, Simpson's rule is less suitable for numerical summation as compared to Hermite's rule [18].

As a matter of fact the purpose of this article is not the computation of π, since many very efficient techniques for this task are known. The main purpose of this article is to introduce the method, which transforms practically useless formula into applicable one. Further, we also wish to show some elementary means in creating the quantitative formulas concerning the rate of convergence of a sequence. This article contains also the message that mathematics and computers do not exclude each other, quite the contrary, they complement one another. This is illustrated by the fact that Mathematica and Maple softwares have the Euler–Maclaurin formula built-in. This is the reason why these programs execute some numerical summations so fast.

2 Transforming the Wallis product

By putting a suitable weights on the terms of Wallis sequence, using the Euler–Maclaurin formula, we can obtain a faster convergent sequence.

By (2a), due to the continuity of the logarithmic function, we have

$$\lim_{n\to\infty} \ln W(n) = \ln W(\infty). \tag{3}$$

But, according to (1),

$$\ln W(n) = \sum_{k=1}^{n} \ln\left(\frac{4k^2}{4k^2-1}\right) \equiv \sum_{k=1}^{n} f(k), \tag{4}$$

where

$$f(x) \equiv \ln\left(\frac{4x^2}{4x^2-1}\right) \equiv \ln\left(1+\frac{1}{4x^2-1}\right) \equiv \ln 4 + 2\ln x - \ln\left(4x^2-1\right), \tag{5a}$$

Function f has the derivatives

$$f'(x) \equiv -\frac{2}{4x^3-x}, \tag{5b}$$

$$f''(x) \equiv 2\frac{12x^2-1}{(4x^3-x)^2}, \tag{5c}$$

and

$$f^{(3)}(x) \equiv -\frac{384x^4-48x^2+4}{(4x^3-x)^3} \equiv -384\frac{\left(x^2-\frac{1}{16}\right)^2+\frac{5}{768}}{(4x^3-x)^3} < 0, \tag{5d}$$

for $x \geq 1$. Because all derivatives of function $f(x)$ converge to 0 as x tends towards infinity we can use the Euler–Maclaurin summation formula of order 1, 2 or 3 for the sum $S(n)$,

$$S(n) := \sum_{k=1}^{n} f(k) \tag{6}$$

figuring in (4). For example, in [17, p. 118], item (23a), there is stated that

$$S(n) = S(m-1) + \frac{1}{2}\left[f(m)+f(n)\right] + \frac{1}{12}\left[f'(n)-f'(m)\right] + \int_m^n f(x)\,dx + \rho_3(m,n), \tag{7a}$$

where, due to [17, p. 117], item (21b),

$$\rho_3(m,n) = -\frac{1}{6}\int_m^n P_3(-x)\; f^{(3)}(x)\,dx. \tag{7b}$$

In (7a) we need also

$$F(x) \equiv \ln\left(\sqrt{\frac{2x-1}{2x+1}}\left(\frac{4x^2}{4x^2-1}\right)^x\right), \tag{7c}$$

being the primitive of f.

From (4)–(7c) we find

$$\ln W(n) = S(n) = S(m) - \frac{f(m)}{2} + \frac{f(n)}{2} + \frac{1}{12}\left[f'(n)-f'(m)\right] + F(n) - F(m) + \rho_3(m,n),$$

hence

$$\ln W_n = C(m) + \left[\frac{1}{2}f(n) + \frac{1}{12}f'(n) + F(n)\right] + \rho_3(m,n), \tag{8}$$

where

$$C(m) \equiv S(m) - \frac{f(m)}{2} - \frac{f'(m)}{12} - F(m)$$

depends only on m. According to (5a), (5b), (7c), and (7b) the equality (8) can be expressed in the form

$$\ln W(n) = C(m) + \frac{1}{2}\ln\left(\frac{4n^2}{4n^2-1}\right) + \frac{1}{12}\left(-\frac{2}{4n^3-n}\right) + \ln\left(\sqrt{\frac{2n-1}{2n+1}}\left(\frac{4n^2}{4n^2-1}\right)^n\right) - \frac{1}{6}\int_m^n P_3(-x)\ f^{(3)}(x)\,dx \tag{9}$$

for $n > m$. As n grows beyond all limits expression $\ln W(n)$ approaches to $\ln W(\infty)$, considering (2a). Further,

$$\lim_{n\to\infty} \ln\left(\sqrt{\frac{2n-1}{2n+1}}\left(\frac{4n^2}{4n^2-1}\right)^n\right) = 0$$

because

$$\left(\frac{4n^2}{4n^2-1}\right)^n \equiv \left[\left(1+\frac{1}{4n^2-1}\right)^{4n^2-1}\right]^{\frac{n}{4n^2-1}}$$

converges towards $e^0 = 1$ as n tends to infinity. Moreover, by [17, p. 115], item (18a), Bernoulli periodic function $P_3(x)$ is bounded

$$|P_3(x)| < \frac{1}{20}, \quad x \in \mathbb{R}. \tag{10}$$

Thus, the integral $\int_m^\infty P_3(-x)\, f^{(3)}(x)dx$ is absolutely convergent due to (5d) and (5c). Hence, letting n to grow beyond all limits in (9), we obtain

$$\ln W(\infty) = C(m) - \frac{1}{6}\int_m^\infty P_3(-x)\ f^{(3)}(x)\,dx\,. \tag{11}$$

Subtracting equation (9) from equation (11) we find the expression

$$\ln\frac{W(\infty)}{W(n)} = -\frac{1}{2}\ln\left(\frac{4n^2}{4n^2-1}\right) + \frac{1}{6\,(4n^3-n)} - \ln\left(\sqrt{\frac{2n-1}{2n+1}}\left(\frac{4n^2}{4n^2-1}\right)^n\right) + \underbrace{\frac{-1}{6}\int_n^\infty P_3(-x)\ f^{(3)}(x)\,dx}_{=\delta_n}. \tag{12a}$$

By (5d) derivative $f^{(3)}$ does not change its sign. Consequently, by the mean value theorem and because of the periodicity of Bernoulli function P_3, there exists, for each integer $n \geq 1$, some $\xi_n \in [0,1]$, such that

$$\int_n^\infty P_3(-x)\, f^{(3)}(x)\,dx = P_3(\xi_n)\int_n^\infty f^{(3)}(x)\,dx = P_3(\xi_n)\cdot\Big(f^{(2)}(\infty) - f^{(2)}(n)\Big) = P_3(\xi_n)\cdot\left[-2\frac{12n^2-1}{(4n^3-n)^2}\right]. \tag{12b}$$

Therefore, the remainder

$$\delta_n := -\frac{1}{6}\int_n^\infty P_3(-x)\ f^{(3)}(x)\,dx$$

in (12a) can be estimated as

$$|\delta_n| \leq \frac{1}{6}\cdot\frac{1}{20}\cdot 2\frac{12n^2-1}{(4n^3-n)^2} < \frac{1}{80\,(n^2-1)^2}.$$

Hence, by (12a) and (12b), there exists, for each integer $n \geq 2$, some $\vartheta_n \in [-1, 1]$, such that

$$W(\infty) = W(n)\left(1 + \frac{1}{2n}\right)\left(1 - \frac{1}{4n^2}\right)^n \exp\left(\frac{1}{6n\,(4n^2-1)} + \frac{\vartheta_n}{80\,(n^2-1)^2}\right) \tag{13a}$$

and consequently

$$W(n) = W(\infty)\left(1 - \frac{1}{2n+1}\right)\left(1 + \frac{1}{4n^2-1}\right)^n \exp\left(\frac{-1}{6n\,(4n^2-1)} - \frac{\vartheta_n}{80\,(n^2-1)^2}\right) \tag{13b}$$

3 Approximating π.

Since $W(\infty) = \frac{\pi}{2}$, by (2a), we nest π, according to (13a), as

$$\pi_1(n) < \pi < \pi_2(n), \tag{14a}$$

valid for any integer $n \geq 2$, where

$$\pi_1(n) := W(n) \cdot \left(2 + \frac{1}{n}\right)\left(1 - \frac{1}{4n^2}\right)^n \exp\left(\frac{1}{6n\,(4n^2-1)} - \frac{1}{80\,(n^2-1)^2}\right) \tag{14b}$$

and

$$\pi_2(n) := \pi_1(n) \cdot \exp\left(\frac{1}{40\,(n^2-1)^2}\right). \tag{14c}$$

Figure 1 illustrates the convergence of sequences $n \mapsto \pi_1(n)$ and $n \mapsto \pi_2(n)$.

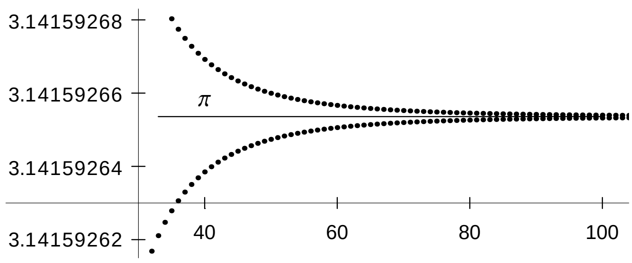

Figure 1. Convergence of sequences $\pi_1(n)$ and $\pi_2(n)$.

According to (1), the Wallis sequence $W(n)$ is increasing monotonically and converges towards $\frac{\pi}{2}$. Thus, for any positive integer n we have $W(n) < \frac{\pi}{2}$. Consequently, from (14b)–(14c), we obtain the estimates

$$\pi_2(n) - \pi_1(n) =$$

$$W(n)\left(2 + \frac{1}{n}\right)\left(1 - \frac{1}{4n^2}\right)^n \exp\left(\frac{1}{6n\,(4n^2-1)} - \frac{1}{80\,(n^2-1)^2}\right)\left[\exp\left(\frac{1}{40\,(n^2-1)^2}\right) - 1\right]$$

$$< \frac{\pi}{2}\left(2 + \frac{1}{2}\right) \cdot 1 \cdot \exp\left(\frac{1}{6 \cdot 2\,(4 \cdot 2^2 - 1)^2}\right) \frac{e}{40\,(n^2-1)^2},$$

that is

$$\pi_2(n) - \pi_1(n) < \frac{3}{10\,(n^2-1)^2}, \tag{14d}$$

for every integer $n \geq 2$. Here we used the simple inequality $e^x < 1+e\cdot x$, valid for $0 < x \leq 1$.

Example 1. By (14d) we estimate $\pi_2(1000)-\pi_1(1000) < 3.1\times 10^{-13}$ and so we expect to obtain twelve decimal places of π by putting $n = 1000$ into (14a)–(14c). Direct computation gives $\pi_1(1000) = 3.14159265358975\ldots$ and $\pi_2(1000) = 3.14159265358983\ldots$, i.e. π is really determined to twelve decimal places as $\pi = 3.141592653589\ldots$.

4 Computing the Wallis products.

Similarly, as we derived (14a), we obtain from (13b) the estimate

$$W_1(n) < W(n) < W_2(n)\,, \tag{15a}$$

where

$$W_1(n) := \frac{\pi}{2}\left(1-\frac{1}{2n+1}\right)\left(1+\frac{1}{4n^2-1}\right)^n \exp\left(\frac{-1}{6n\,(4n^2-1)}-\frac{1}{80\,(n^2-1)^2}\right) \tag{15b}$$

and

$$W_2(n) := \frac{\pi}{2}\left(1-\frac{1}{2n+1}\right)\left(1+\frac{1}{4n^2-1}\right)^n \exp\left(\frac{-1}{6n\,(4n^2-1)}+\frac{1}{80\,(n^2-1)^2}\right) \tag{15c}$$

for any integer $n \geq 2$. From these relations we can compute $W(n)$ for rather large n. However, for a very large n a direct computation using these formulas is not easy due to the third factor in (15b)–(15c), which varies like $\exp\left(\frac{n}{4n^2-1}\right)$, according to the well known convergence towards the exponent function. The rate of this convergence is evident from the following lemma:

Lemma 1 *For any positive real x and $t \geq 2x$ there holds the estimate*

$$\exp\left(x-\frac{x^2}{2t}\right) \;<\; \left(1+\frac{x}{t}\right)^t \;<\; \exp\left(x-\frac{x^2}{3t}\right).$$

Indeed, integrating the inequality

$$1-t \;<\; \frac{1}{1+t} \;<\; 1-\frac{2}{3}\,t$$

valid for $t \in (0,\frac{1}{2})$, we obtain the estimate

$$y-\frac{y^2}{2} \;<\; \int_0^y \frac{dt}{1+t} \;=\; \ln(1+y) \;<\; y-\frac{y^2}{3}$$

valid for $y \in (0,\frac{1}{2}]$. Because for any $x > 0$ and $t \geq 2x$ the number $y := \frac{x}{t}$ lies in the interval $(0,\frac{1}{2}]$, we can put this y into the relations above to obtain

$$\frac{x}{t}-\frac{x^2}{2t^2} \;<\; \ln\left(1+\frac{x}{t}\right) \;<\; \frac{x}{t}-\frac{x^2}{3t^2},$$

which verifies the proposition.

According to the just verified lemma, where we put $t = n$ and $x = \frac{n}{4n^2-1}$, we find that

$$\exp\left(\tfrac{n}{4n^2-1}-\tfrac{n}{2(4n^2-1)^2}\right) < \left(1+\frac{1}{4n^2-1}\right)^n < \exp\left(\tfrac{n}{4n^2-1}-\tfrac{n}{3(4n^2-1)^2}\right) \tag{16}$$

for every integer $n \geq 1$.

From (15a)–(16) we deduce the estimate

$$W_1^*(n) < W(n) < W_2^*(n)\,, \tag{17a}$$

where

$$W_1^*(n) := \frac{\pi}{2}\left(1-\frac{1}{2n+1}\right)\exp\left(\varphi(n)\right), \tag{17b}$$

$$W_2^*(n) := \frac{\pi}{2}\left(1-\frac{1}{2n+1}\right)\exp\left(\psi(n)\right), \tag{17c}$$

$$\varphi(n) := \frac{n}{4n^2-1}-\frac{n}{2\left(4n^2-1\right)^2}-\frac{1}{6n\left(4n^2-1\right)}-\frac{1}{80\left(n^2-1\right)^2}, \tag{17d}$$

and

$$\psi(n) = \frac{n}{4n^2-1}-\frac{n}{3\left(4n^2-1\right)^2}-\frac{1}{6n\left(4n^2-1\right)}+\frac{1}{80\left(n^2-1\right)^2} \tag{17e}$$

for any integer $n \geq 2$. For such n we have

$$\frac{0.9}{4n} < \varphi(n) < \psi(n) < \frac{1.02}{4n} \tag{17f}$$

and for $n \geq 18$ there holds the estimate

$$\frac{0.999}{4n} < \varphi(n) < \psi(n) < \frac{1.00001}{4n}. \tag{17g}$$

Indeed, from (17d) there follows

$$\begin{aligned}\varphi(n) &= \frac{n}{4n^2-1}\left[1-\frac{1}{2\left(4n^2-1\right)}-\frac{1}{6n^2}\right]-\frac{1}{4\left(n+1\right)\left[20\left(n-1\right)^2\left(n+1\right)\right]}\\ &> \frac{1}{4n}\underbrace{\left[1-\frac{1}{2\left(4n^2-1\right)}-\frac{1}{6n^2}-\frac{1}{20\left(n-1\right)^2\left(n+1\right)}\right]}_{=\lambda(n)} = \frac{1}{4n}\cdot\lambda(n)\end{aligned}$$

and from (17e) we find

$$\begin{aligned}\psi(n) &< \frac{n}{4n^2-1}+\frac{1}{80\left(n^2-1\right)^2} < \frac{n}{4n^2}+\frac{1}{4\left(n+1\right)\left[20\left(n-1\right)^2\left(n+1\right)\right]}\\ &< \frac{1}{4n}\underbrace{\left[1+\frac{1}{20\left(n-1\right)^2\left(n+1\right)}\right]}_{=\mu(n)} = \frac{1}{4n}\cdot\mu(n).\end{aligned}$$

Factor $\lambda(n)$ increases and $\mu(n)$ decreases, hence $\lambda(n) \geq \lambda(2) > 0.9$ and $\mu(n) \leq \mu(2) < 1.02$ for $n \geq 2$, moreover $\lambda(n) \geq \lambda(18) > 0.999$ and $\mu(n) \leq \mu(18) < 1.00001$ for $n \geq 18$. This confirms (17f) and (17g).

In order to simplify (17a)–(17e), we use the Taylor's formula of order 1 for the exponential function, to produce the estimate

$$1+x < e^x < 1+\left(1+\frac{e^x}{2}x\right)x, \tag{18a}$$

valid for positive x. Thus, by (17f), we have

$$\exp\left(\varphi(n)\right) > 1+\varphi(n) \tag{18b}$$

and

$$\exp\left(\psi(n)\right) < 1+\omega(\psi(n))\cdot\psi(n), \tag{18c}$$

where $\omega(x) := 1 + e^x x/2$ increases for $x > 0$. Thus, due to (17f)–(17g), there holds the estimate

$$\omega(\psi(n)) < \omega\left(\frac{1.02}{4n}\right) \leq \omega\left(\frac{1.02}{4\cdot 2}\right) < 1.073$$

for $n \geq 2$ and

$$\omega(\psi(n)) < \omega\left(\frac{1.00001}{4n}\right) \leq \omega\left(\frac{1.0001}{4\cdot 18}\right) < 1.0071$$

for $n \geq 18$. Hence, according to (18c), we obtain

$$\exp\left(\psi(n)\right) < 1 + 1.073\cdot\psi(n) \tag{18d}$$

for $n \geq 2$ and

$$\exp\left(\psi(n)\right) < 1 + 1.0071\cdot\psi(n) \tag{18e}$$

for $n \geq 18$.

Considering (17a)–(17c) and (18b), (18c) we find

$$a(n) < W(n) < b(n), \tag{19a}$$

where, for $n \geq 2$,

$$a(n) := \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right)(1 + \varphi(n)) \tag{19b}$$

and

$$b(n) := \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right)(1 + 1.073\psi(n))\,, \tag{19c}$$

but for $n \geq 18$ we can take more accurate bound

$$b(n) := \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right)(1 + 1.0071\psi(n))\ . \tag{19d}$$

Figure 2 shows the graph of sequence $n \mapsto b(n) - a(n)$ for $18 \leq n \leq 84$.

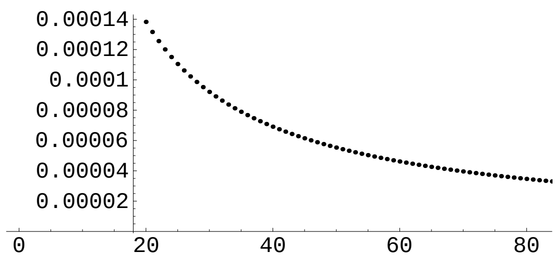

Figure 2. Graph of sequence $n \mapsto b(n) - a(n)$.

5 Estimating the rate of convergence of the Wallis products

By (19a)–(19d) and (17f)–(17g) we obtain

$$a(n) > \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right)\left(1 + \frac{1+\delta_1}{4n}\right) = \frac{\pi}{2}\left(1 - \frac{1-\delta_1}{2(2n+1)}\right)$$

and

$$b(n) < \frac{\pi}{2}\left(1 - \frac{1}{2n+1}\right)\left(1 + \frac{1+\delta_2}{4n}\right) = \frac{\pi}{2}\left(1 - \frac{1-\delta_2}{2(2n+1)}\right),$$

where we can take $\delta_1 = -0.1$ and $\delta_2 = 0.1$ for $n \geq 2$ and similarly $\delta_1 = -0.001$ and $\delta_2 = 0.008$ for $n \geq 18$. Consequently

$$a(n) \;>\; \frac{\pi}{2}\left(1-\frac{1.1}{2(2n+1)}\right) \;>\; \frac{\pi}{2}\left(1-\frac{1.1}{4n}\right)$$

for $n \geq 2$, and

$$a(n) \;>\; \frac{\pi}{2}\left(1-\frac{1.001}{2(2n+1)}\right) \;>\; \frac{\pi}{2}\left(1-\frac{1.001}{4n}\right)$$

for $n \geq 18$. Similarly, for $n \geq 2$, we have

$$b(n) \;<\; \frac{\pi}{2}\left(1-\frac{0.9}{2(2n+1)}\right) \underbrace{\;<\;}_{n\geq 4} \frac{\pi}{2}\left(1-\frac{0.8}{4n}\right)$$

and for $n \geq 18$ we obtain

$$b(n) \;<\; \frac{\pi}{2}\left(1-\frac{0.992}{2(2n+1)}\right) \underbrace{\;<\;}_{n\geq 250} \frac{\pi}{2}\left(1-\frac{0.99}{4n}\right).$$

Therefore, according to (19a), using also direct computations, we find

$$\frac{\pi}{2}\left(1-\frac{1.1}{4n}\right) < W(n) < \frac{\pi}{2}\left(1-\frac{0.8}{4n}\right) \tag{20a}$$

for $n \geq 3$, and

$$\frac{\pi}{2}\left(1-\frac{1.001}{4n}\right) < W(n) < \frac{\pi}{2}\left(1-\frac{0.990}{4n}\right) \tag{20b}$$

for $n \geq 62$. This way we have estimated the rate of linear convergence $\lim\limits_{n\to\infty} W(n) = \frac{\pi}{2}$:

$$\pi \cdot \frac{0.990}{8n} < \frac{\pi}{2} - W(n) < \pi \cdot \frac{1.001}{8n} \tag{20c}$$

for every integer $n \geq 62$. Consequently, the estimate

$$\frac{0.3}{n} < \frac{\pi}{2} - W(n) < \frac{0.4}{n} \tag{20d}$$

is true even for $n \geq 3$, by using direct computation [26]. The bounds could certainly be improved simply by using the Euler–Maclaurin formula of order higher than 3. In the literature better bounds are also known, see for example [9] and [15].

Example 2. According to (20c) we have $\frac{\pi}{2} - 4\cdot 10^{-3001} < W(1000^{1000}) < \frac{\pi}{2} - 3\cdot 10^{-3001}$.

Considering (14d), we see that the convergence of sequence $n \mapsto 2W(n)$ towards π is much slower than the convergence of sequences $n \mapsto \pi_1(n)$ and $n \mapsto \pi_2(n)$. Modified Wallis sequences $\frac{1}{2}\,\pi_1(n)$ and $\frac{1}{2}\,\pi_2(n)$ have considerably accelerated convergence, relatively to the convergence of the original Wallis sequence. Figure 3 illustrates relation (20a) by showing the graph of sequence $n \mapsto \frac{\pi}{2} - W(n)$ and the graphs of functions $n \mapsto \pi\frac{0.8}{8n}$, and $n \mapsto \pi\frac{1.1}{8n}$ for $3 \leq n \leq 60$.

Remarks.

R1. If we should approximate the Wallis product $W(n) = \frac{1}{2n+1}\left(\prod\limits_{k=1}^{n} \frac{2k}{2k-1}\right)^2$ by using the Euler–Maclaurin summation formula of order 3 for the function

$$g(x) \equiv \ln\left(\frac{2x}{2x-1}\right) \equiv \ln 2 + \ln x - \ln\left(2x-1\right), \tag{21}$$

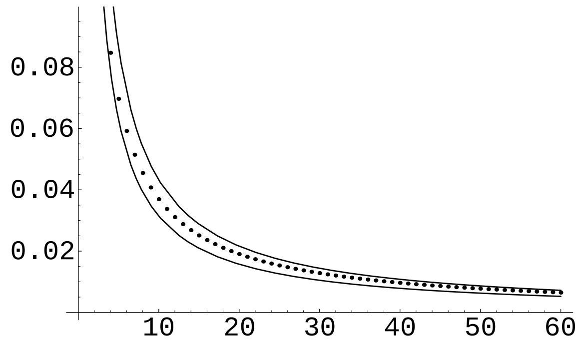

Figure 3. Lower and upper bounds $n \mapsto \pi \frac{0.8}{8n}$ and $n \mapsto \pi \frac{1.1}{8n}$ for the null-sequence $n \mapsto \frac{\pi}{2} - W(n)$.

we should obtain the expression

$$\pi = W(n) \cdot \left(2 + \frac{1}{n}\right) \cdot e \left(1 - \frac{1}{2n}\right)^{2n} \exp\left(\frac{1}{6n(2n-1)}\right) \cdot \exp\left(\frac{\vartheta_n}{60n(n-1)^2}\right)$$

for some $\vartheta_n \in (-1, 1)$. This formula is a little bit better than (2c), but less accurate than (13a).

R2. For the functions $f(x)$ and $g(x)$ from (5a) and (21) we should use the Euler–Maclaurin formula of the higher order than the order three to obtain more accurate formula for π and $W(n)$. However, such formulas should be more complicated than the one just presented.

R3. In 1665[3] , ten years after Wallis, Newton made the very first application of his new calculus. He computed π to sixteen decimal places using the binomial expansion

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \left[\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \left(-t^2\right)^k\right] dt = \sum_{k=0}^{\infty} \int_0^x \binom{-\frac{1}{2}}{k} (-1)^k t^{2k}\, dt$$
$$= x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \cdots,$$

where he put $x = \frac{1}{2}$ to obtain the expansion for $\frac{\pi}{6}$. Later, in 1666[4] or 1671[5] or 1680[6], he found another way to compute sixteen correct digits of π, using his formula[7] employing 22 terms of an infinite series expansion:

$$\pi = \frac{3\sqrt{3}}{4} + 24 \int_0^{1/4} \sqrt{x}\,\sqrt{1-x}\, dx$$
$$= \frac{3\sqrt{3}}{4} + 24 \left(\frac{1}{12} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} \cdots\right).$$

After a 15 digits computation, during the plague in Cambridge and London (between 1665 and 1666)[8] Newton wrote: I am ashamed to tell you to how many figures I carried these computations, having no other business at the time [5]. Perhaps Newton should have less

[3] http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Pi_chronology.html#s44; http://wvwv.essortment.com/pimathematicsa_rjar.htm; http://www.geocities.com/lady_lizzie/info.html

[4] http://members.tripod.com/egyptonline/newton.htm

[5] http://numbers.computation.free.fr/Constants/Pi/piCompute.html

[6] http://www.colab.sfu.ca/PiDay/3_14/Pi1.html; author Greg Fee

[7] See [5, pp. 110&111] or [10, pp. 174–177].

[8] http://people.bath.ac.uk/ma3mju/calc.html#newt

work in his computation if the Euler–Maclaurin method of summation should be known to him. However this method has been discovered about 60 years later and unfortunately not in its correct form – with a remainder, which has been presented the first time by Poisson in 1823 [21].

R4. Knowing the Euler–Maclaurin formula it is not difficult to find numerical sum of slowly convergent Leibniz series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{\pi}{4},$$

which, like the Wallis product, also is considered in the literature as unsuitable for numerical computation of π.

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