A difficult limit

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Let $u(m,n) = \sum_{k=0}^{n} \frac{\binom{n}{k}}{m+k}$, $v(m,n) = \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{m+k}$ and $w(m,n) = \frac{u(m,n)}{v(m,n)}$. It is shown in [1] that, surprisingly, $w(m,n)$ is an integer. Let $c(n) = w(n,n)/\left(2^n \binom{2n}{n}\right)$. It is shown in [1] that

$$c(n) \to \frac{2}{3} \quad \text{as} \quad n \to \infty. \tag{1}$$

The object of this note is to give an alternative route to (1) and to give an asymptotic expansion for $c(n)$.

In order to get a lower bound for $c(n)$, we start with the formula, given in [1],

$$c(n) = 2^{-n} n \int_0^1 x^{n-1}(1+x)^n dx.$$

We have

$$\begin{aligned} C(q) = \sum_{n \geq 0} c(n) q^n &= \sum_{n \geq 1} 2^{-n} n q^n \int_0^1 x^{n-1}(1+x)^n dx \\ &= \int_0^1 \sum_{n \geq 1} 2^{-n} n x^{n-1}(1+x)^n q^n \\ &= \int_0^1 \frac{1}{2}(1+x)q \sum_{n \geq 0} n \left(\frac{x(1+x)q}{2}\right)^{n-1} dx \\ &= \int_0^1 \frac{1}{2}(1+x)q \frac{1}{\left(1 - \frac{x(1+x)q}{2}\right)^2} dx \\ &= \int_0^1 \frac{2(1+x)q}{(2-qx-qx^2)^2} dx. \end{aligned}$$

This is a standard integral, and the editors tell me that MATHEMATICA can evaluate it. MAPLE 8 cannot, and MAPLE 9 gives me something that I should be able to massage into what I get by hand, which is, with $s = \sqrt{8q+q^2}$,

$$\begin{aligned} C(q) &= \frac{5q+q^2}{(8+q)(1-q)} + \frac{2q^2}{s^3} \log \frac{1 + \frac{s}{4-q}}{1 - \frac{s}{4-q}} \\ &= \frac{5q+q^2}{(8+q)(1-q)} + \frac{2q^2}{s^3} \left(2\frac{s}{4-q} + \frac{2}{3}\left(\frac{s}{4-q}\right)^3 + \frac{2}{5}\left(\frac{s}{4-q}\right)^5 + \cdots \right) \\ &= \frac{5q+q^2}{(8+q)(1-q)} + \frac{4q}{(8+q)(4-q)} + \frac{4q^2}{3(4-q)^3} + \frac{4q^3(8+q)}{5(4-q)^5} + \frac{4q^4(8+q)^2}{7(4-q)^7} + \cdots \\ &= \frac{q(3-q)}{(4-q)(1-q)} + \frac{4q^2}{3(4-q)^3} + \frac{4q^3(8+q)}{5(4-q)^5} + \frac{4q^4(8+q)^2}{7(4-q)^7} + \cdots . \end{aligned}$$

Hence

$$C(q) > \frac{q(3-q)}{(4-q)(1-q)}$$

and

$$c(n) \geq \frac{2}{3} + \frac{1}{3}\left(\frac{1}{4}\right)^n.$$

To obtain an upper bound for $c(n)$, we start with the formula, also given in [1],

$$w(m,n) = \sum_{k=0}^{n} \binom{m+n}{m+k}\binom{m+k-1}{k}.$$

We have

$$\begin{aligned}
\sum_{n\geq 0} w(m,n)y^n &= \sum_{n\geq 0}\sum_{k=0}^{n} \binom{m+n}{m+k}\binom{m+k-1}{k} y^n \\
&= \sum_{k\geq 0}\sum_{n=k}^{\infty} \binom{m+n}{m+k}\binom{m+k-1}{k} y^n \\
&= \sum_{k\geq 0}\sum_{n\geq 0} \binom{m+k+n}{m+k}\binom{m+k-1}{k} y^{k+n} \\
&= \sum_{k\geq 0} \binom{m+k-1}{k} y^k (1-y)^{-(m+k+1)} \\
&= (1-y)^{-(m+1)} \sum_{k\geq 0} \binom{m+k-1}{m-1}\left(\frac{y}{1-y}\right)^k \\
&= (1-y)^{-(m+1)} \left(1 - \frac{y}{1-y}\right)^{-m} \\
&= (1-y)^{-1}(1-2y)^{-m} \\
&= (1+y+y^2+\ \cdots)\left(1 + \binom{m}{m-1}2y + \binom{m+1}{m-1}2^2y^2 + \cdots\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
w(n,n) &= 2^n\binom{2n-1}{n-1} + 2^{n-1}\binom{2n-2}{n-1} + \cdots\ + 1 \\
&= \sum_{k\geq 0} 2^{n-k}\binom{2n-1-k}{n-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
c(n) &= \sum_{k\geq 0} 2^{n-k}\binom{2n-1-k}{n-1} \Big/ 2^n\binom{2n}{n} \\
&= \sum_{k\geq 0} 2^{-k}\frac{(2n-1-k)!}{(n-1)!(n-k)!}\cdot\frac{n!n!}{(2n)!} \\
&= \sum_{k\geq 0} 2^{-(2k+1)}\frac{(2n)(2n-2)\cdots(2n-2k+2)}{(2n-1)(2n-2)\cdots(2n-k)}.
\end{aligned}$$

It follows that

$$c(n) < \frac{1}{2} + \frac{2n}{2n-1}\sum_{k\geq 1} 2^{-(2k+1)} = \frac{1}{2} + \frac{1}{6}\cdot\frac{2n}{2n-1} = \frac{2}{3} + \frac{1}{6(2n-1)}.$$

In summary,

$$\frac{2}{3} + \frac{1}{3 \times 4^n} \le c(n) < \frac{2}{3} + \frac{1}{6(2n-1)}.$$

So (1) holds.

But much more is true! We have

$$\begin{aligned} c(n) &= \frac{1}{2} + \frac{1}{8} \cdot \frac{2n}{2n-1} + \frac{1}{32} \frac{2n(2n-2)}{(2n-1)(2n-2)} + \frac{1}{128} \frac{2n(2n-2)(2n-4)}{(2n-1)(2n-2)(2n-3)} + \cdots \\ &= \frac{1}{2} + \frac{1}{8} \left(1 + \frac{\frac{1}{2}}{n} + \cdots\right) + \frac{1}{32} \left(1 + \frac{\frac{1}{2}}{n} + \cdots\right) + \frac{1}{128} \left(1 + \frac{0}{n} + \cdots\right) + \frac{1}{512} \left(1 - \frac{1}{n} + \cdots\right) + \cdots \\ &= \frac{2}{3} + \frac{2}{27n} + \cdots \end{aligned}$$

after some calculation. Indeed,

$$c(n) = \frac{2}{3} + \frac{2}{27n} + \frac{2}{81n^2} - \frac{2}{729n^3} - \frac{110}{6561n^4} - \frac{1459}{69984n^5} + \frac{13493}{9447840n^6} \cdots .$$

References

[1] A. Nuijenhuis and R. Stong, *Solution to problem 10886 (a) and (c)*, American Mathematical Monthly **110** (2003), 344–345.

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