The asymptotic solution of a difference equation considered by Ramanujan

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1 Introduction

In the last year of his life, while he was dying in India, Ramanujan managed to produce some startling results, which he duly recorded, mostly without any indication of derivation, as was his wont. These results, about 600 of them, have come down to us in a manuscript dubbed by George E. Andrews the "Lost Notebook". This "Lost Notebook" was reproduced [1] in 1988, for the Centenary of Ramanujan's birth. Since Andrews' discovery of the LNB in 1976, he and latterly Bruce C. Berndt have been working on editing it, with a view to publication. I have recently been privileged to see a draft version of the Springer book (at least the first volume of the projected four volume work). I have been inspired to solve one problem in my own way (though I owe the first steps to Andrews and Berndt). This note is the result.

In Chapter 8 of their book, Andrews and Berndt gather together results of Ramanujan concerning the continued fraction

$$\frac{1}{1} - \frac{1}{1+q} - \frac{1}{1+q^2} - \cdots - \frac{1}{1+q^{N-1}+a}$$

and related material. They present and give a proof of Ramanujan's result that this continued fraction has three different limits as $N \to \infty$, according to the residue class of N modulo 3. They describe this result as one of the most fascinating in the Lost Notebook. In the same chapter they present Ramanujan's study of the behaviour of the continued fraction

$$u_\lambda = \frac{1}{1+q^{\lambda+1}} - \frac{1}{1+q^{\lambda+2}} - \cdots$$

as $q \to 1$, where he replaces q by e^x and considers $x \to 0$. He finds an expression for u_λ as a series in powers of x. I shall tackle the same problem. In reading this, I invite you to realise that this problem represents perhaps half a day's work of a dying Ramanujan. It took me a week, on and off, with the benefit of MAPLE to avoid doing a partial fractions expansion by hand, and to check my working.

I shall carry out the calculations as far as the term in x^2, as does Ramanujan, and although my solution "looks different" from Ramanujan's, the two are equivalent.

With u_λ defined above, we have

$$u_\lambda + \frac{1}{u_{\lambda-1}} = 1 + e^{\lambda x}.$$

Let $v_\lambda = 1/(u_\lambda - 1)$. It follows that

$$v_\lambda = \frac{v_{\lambda-1}+1}{1+(e^{\lambda x}-1)(v_{\lambda-1}+1)}.$$

If we set $x = 0$, we find $v_\lambda = v_{\lambda-1} + 1$, of which the solution is $v_\lambda = \lambda + c$ where c is arbitrary, and

$$u_\lambda = 1 + \frac{1}{v_\lambda} = 1 + \frac{1}{\lambda+c} = \frac{\lambda+c+1}{\lambda+c}.$$

Now suppose

$$u_\lambda = \frac{\lambda+c+1}{\lambda+c} + a(\lambda)x + \cdots .$$

Then

$$\begin{aligned} u_{\lambda-1} &= \frac{\lambda+c}{\lambda+c-1} + a(\lambda-1)x + \cdots \\ &= \frac{\lambda+c}{\lambda+c-1}\left(1 + \frac{\lambda+c-1}{\lambda+c}a(\lambda-1)x + \cdots\right) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{u_{\lambda-1}} &= \frac{\lambda+c-1}{\lambda+c}\left(1 - \frac{\lambda+c-1}{\lambda+c}a(\lambda-1)x + \cdots\right) \\ &= \frac{\lambda+c-1}{\lambda+c} - \left(\frac{\lambda+c-1}{\lambda+c}\right)^2 a(\lambda-1)x + \cdots . \end{aligned}$$

It follows that

$$u_\lambda + \frac{1}{u_{\lambda-1}} = 2 + \left\{a(\lambda) - \left(\frac{\lambda+c-1}{\lambda+c}\right)^2 a(\lambda-1)\right\}x + \cdots .$$

Thus we want to find $a(\lambda)$ to satisfy

$$a(\lambda) - \left(\frac{\lambda+c-1}{\lambda+c}\right)^2 a(\lambda-1) = \lambda.$$

This can be written

$$(\lambda+c)^2 a(\lambda) - (\lambda+c-1)^2 a(\lambda-1) = \lambda(\lambda+c)^2 = \lambda^3 + 2c\lambda^2 + c^2\lambda.$$

Summing on λ gives

$$(\lambda+c)^2 a(\lambda) = \frac{1}{4}\lambda^2(\lambda+1)^2 + \frac{c}{3}\lambda(\lambda+1)(2\lambda+1) + \frac{c^2}{2}\lambda(\lambda+1) + c^2\alpha$$

where $\alpha = a(0)$ is arbitrary. So

$$a(\lambda) = \frac{\frac{1}{4}\lambda^2(\lambda+1)^2 + \frac{c}{3}\lambda(\lambda+1)(2\lambda+1) + \frac{c^2}{2}\lambda(\lambda+1) + c^2\alpha}{(\lambda+c)^2}.$$

Thus far we have

$$u_\lambda = \frac{\lambda+c+1}{\lambda+c} + \frac{\frac{1}{4}\lambda^2(\lambda+1)^2 + \frac{c}{3}\lambda(\lambda+1)(2\lambda+1) + \frac{c^2}{2}\lambda(\lambda+1) + c^2\alpha}{(\lambda+c)^2}x + \cdots .$$

Now suppose

$$u_\lambda = \frac{\lambda+c+1}{\lambda+c} + \frac{\frac{1}{4}\lambda^2(\lambda+1)^2 + \frac{c}{3}\lambda(\lambda+1)(2\lambda+1) + \frac{c^2}{2}\lambda(\lambda+1) + c^2\alpha}{(\lambda+c)^2}x + b(\lambda)x^2 + \cdots .$$

Then

$$u_{\lambda-1} = \frac{\lambda+c}{\lambda+c-1} + \frac{\frac14(\lambda-1)^2\lambda^2 + \frac{c}{3}\lambda(\lambda-1)(2\lambda-1) + \frac{c^2}{2}(\lambda-1)\lambda + c^2\alpha}{(\lambda+c-1)^2}x$$
$$+ b(\lambda-1)x^2 + \cdots$$
$$= \frac{\lambda+c}{\lambda+c-1}\Bigg(1 + \frac{\frac14(\lambda-1)^2\lambda^2 + \frac{c}{3}\lambda(\lambda-1)(2\lambda-1) + \frac{c^2}{2}(\lambda-1)\lambda + c^2\alpha}{(\lambda+c)(\lambda+c-1)}x$$
$$+ \frac{\lambda+c-1}{\lambda+c}b(\lambda-1)x^2 + \cdots\Bigg)$$

and

$$\frac{1}{u_{\lambda-1}} = \frac{\lambda+c-1}{\lambda+c}\Bigg(1 - \frac{\frac14(\lambda-1)^2\lambda^2 + \frac{c}{3}\lambda(\lambda-1)(2\lambda-1) + \frac{c^2}{2}(\lambda-1)\lambda + c^2\alpha}{(\lambda+c)(\lambda+c-1)}x$$
$$+ \Bigg[\frac{(\frac14(\lambda-1)^2\lambda^2 + \frac{c}{3}\lambda(\lambda-1)(2\lambda-1) + \frac{c^2}{2}(\lambda-1)\lambda + c^2\alpha)^2}{(\lambda+c)^2(\lambda+c-1)^2}$$
$$- \frac{\lambda+c-1}{\lambda+c}b(\lambda-1)\Bigg]x^2 + \cdots\Bigg)$$
$$= \frac{\lambda+c-1}{\lambda+c} - \frac{\frac14(\lambda-1)^2\lambda^2 + \frac{c}{3}\lambda(\lambda-1)(2\lambda-1) + \frac{c^2}{2}(\lambda-1)\lambda + c^2\alpha}{(\lambda+c)^2}x$$
$$+ \Bigg[\frac{(\frac14(\lambda-1)^2\lambda^2 + \frac{c}{3}\lambda(\lambda-1)(2\lambda-1) + \frac{c^2}{2}(\lambda-1)\lambda + c^2\alpha)^2}{(\lambda+c)^3(\lambda+c-1)}$$
$$- \left(\frac{\lambda+c-1}{\lambda+c}\right)^2 b(\lambda-1)\Bigg]x^2 + \cdots.$$

It follows that

$$u_\lambda + \frac{1}{u_{\lambda-1}} = 2 + \lambda x + \Bigg[b(\lambda) + \frac{(\frac14(\lambda-1)^2\lambda^2 + \frac{c}{3}\lambda(\lambda-1)(2\lambda-1) + \frac{c^2}{2}(\lambda-1)\lambda + c^2\alpha)^2}{(\lambda+c)^3(\lambda+c-1)}$$
$$- \left(\frac{\lambda+c-1}{\lambda+c}\right)^2 b(\lambda-1)\Bigg]x^2 + \cdots.$$

We want to find $b(\lambda)$ to satisfy

$$b(\lambda) + \frac{(\frac14(\lambda-1)^2\lambda^2 + \frac{c}{3}\lambda(\lambda-1)(2\lambda-1) + \frac{c^2}{2}(\lambda-1)\lambda + c^2\alpha)^2}{(\lambda+c)^3(\lambda+c-1)}$$
$$- \left(\frac{\lambda+c-1}{\lambda+c}\right)^2 b(\lambda-1) = \frac{\lambda^2}{2},$$

or,

$$144(\lambda+c)^2 b(\lambda) - 144(\lambda+c-1)^2 b(\lambda-1)$$
$$= 72\lambda^2(\lambda+c)^2 - \frac{(3(\lambda-1)^2\lambda^2 + 4c\lambda(\lambda-1)(2\lambda-1) + 6c^2(\lambda-1)\lambda + 12c^2\alpha)^2}{(\lambda+c)(\lambda+c-1)}.$$

If we convert the right hand side to partial fractions with respect to c, we can write the last equation as

$$\begin{aligned}
&144(\lambda+c)^2 b(\lambda) - 144(\lambda+c-1)^2 b(\lambda-1)\\
&\quad= (-36\lambda^4+72\lambda^3+36\lambda^2-144\alpha\lambda^2+144\alpha\lambda-144\alpha^2)c^2\\
&\qquad+(-24\lambda^5+60\lambda^4+96\lambda^3+12\lambda^2+96\alpha\lambda^3-144\alpha\lambda^2+48\alpha\lambda+288\alpha^2\lambda-144\alpha^2)c\\
&\qquad+(-16\lambda^6+48\lambda^5+20\lambda^4+24\lambda^3-4\lambda^2-120\alpha\lambda^4+240\alpha\lambda^3-168\alpha\lambda^2+48\alpha\lambda\\
&\qquad-432\alpha^2\lambda^2+432\alpha^2\lambda-144\alpha^2)\\
&\qquad+\frac{\lambda^4(\lambda^4-2\lambda^2+1+24\alpha\lambda^2-24\alpha+144\alpha^2)}{\lambda+c}\\
&\qquad-\frac{(\lambda-1)^4((\lambda-1)^4-2(\lambda-1)^2+1+24\alpha(\lambda-1)^2-24\alpha+144\alpha^2)}{\lambda+c-1}.
\end{aligned}$$

If we sum on λ we find

$$\begin{aligned}
&5040(\lambda+c)^2 b(\lambda) - 5040c^2\beta\\
&\quad= (-252\lambda^5+1260\lambda^3+1260\lambda^2+252\lambda-1680\alpha\lambda^3+1680\alpha\lambda-5040\alpha^2\lambda)c^2\\
&\qquad+(-140\lambda^6+1540\lambda^4+2520\lambda^3+1120\lambda^2+840\alpha\lambda^4-840\alpha\lambda^2+5040\alpha^2\lambda^2)c\\
&\qquad+(-80\lambda^7+700\lambda^5+1260\lambda^4+700\lambda^3-60\lambda+840\alpha\lambda^3-840\alpha\lambda-5040\alpha^2\lambda^2)\\
&\qquad+\frac{35\lambda^4(\lambda^4-2\lambda^2+1+24\alpha\lambda^2-24\alpha+144\alpha^2)}{\lambda+c}
\end{aligned}$$

where $\beta = b(0)$ is arbitrary. It follows that

$$\begin{aligned}
b(\lambda) &= \frac{1}{5040(\lambda+c)^3}\Big((-45\lambda^8+630\lambda^6+1260\lambda^5+735\lambda^4-60\lambda^2)\\
&\quad+(-220\lambda^7+2240\lambda^5+3780\lambda^4+1820\lambda^3-60\lambda)c\\
&\quad+(-392\lambda^6+2800\lambda^4+3780\lambda^3+1372\lambda^2-840\alpha\lambda^4+840\alpha\lambda^2+5040\beta\lambda)c^2\\
&\quad+(-252\lambda^5+1260\lambda^3+1260\lambda^2+252\lambda-1680\alpha\lambda^3+1680\alpha\lambda-5040\alpha^2\lambda+5040\beta)c^3\Big)\\
&= -\frac{1}{5040(\lambda+c)^3}\Big\{\lambda(\lambda+1)\Big(15\lambda(\lambda+2)(3\lambda^4-9\lambda^3-21\lambda^2-3\lambda+2)\\
&\quad+20(11\lambda^5-11\lambda^4-101\lambda^3-88\lambda^2-3\lambda+3)c\\
&\quad+28\lambda(14\lambda^3-14\lambda^2-86\lambda-49+30\alpha\lambda-30\alpha)c^2\\
&\quad+84(3\lambda^3-3\lambda^2-12\lambda-3+20\alpha\lambda-20\alpha)c^3\Big)\\
&\quad-5040\beta\lambda c^2+(5040\alpha^2\lambda-5040\beta)c^3\Big\}.
\end{aligned}$$

Thus we have

$$
\begin{aligned}
u_\lambda = \frac{\lambda+c+1}{\lambda+c} &+ \frac{1}{12(\lambda+c)^2}\Big\{\lambda(\lambda+1)\Big(3\lambda(\lambda+1)+(8\lambda+4)c+6c^2\Big)+12\alpha c^2\Big\}x \\
&- \frac{1}{5040(\lambda+c)^3}\Big\{\lambda(\lambda+1)\Big(15\lambda(\lambda+2)(3\lambda^4-9\lambda^3-21\lambda^2-3\lambda+2) \\
&\qquad + 20(11\lambda^5-11\lambda^4-101\lambda^3-88\lambda^2-3\lambda+3)c \\
&\qquad + 28\lambda(14\lambda^3-14\lambda^2-86\lambda-49+30\alpha\lambda-30\alpha)c^2 \\
&\qquad + 84(3\lambda^3-3\lambda^2-12\lambda-3+20\alpha\lambda-20\alpha)c^3\Big) \\
&\qquad - 5040\beta\lambda c^2+(5040\alpha^2\lambda-5040\beta)c^3\Big\}x^2+\ \cdots\ .
\end{aligned}
$$

Ramanujan's formulation is

$$
\begin{aligned}
u_\lambda = 1-\frac{\phi_0}{1-\lambda\phi_0} &+ x\left(\frac{\lambda+1}{2}+\frac{\phi_1+(\lambda^2-1)(\frac12-\frac23\lambda\phi_0+\frac14\lambda^2\phi_0^2)}{(1-\lambda\phi_0)^2}\right) \\
&+ x^2\left\{\frac{\lambda(\lambda+1)(\lambda+2)}{12}-\frac{\phi_2+\lambda(\lambda^2-1)(\lambda^2-4)\left(\frac{1}{45}-\frac{\lambda\phi_0}{36}+\frac{(\lambda^2+\frac13)\phi_0^2}{112}\right)}{(1-\lambda\phi_0)^2}\right. \\
&\qquad \left. -\frac{\lambda}{(1-\lambda\phi_0)^3}\left(\phi_1+\frac{\lambda^2-1}{6}(1-\frac12\lambda\phi_0)\right)^2\right\}+\ \cdots\ .
\end{aligned}
$$

Ramanujan's formulation is equivalent to mine, with his ϕ_0, ϕ_1, ϕ_2 related to my α, β, c by

$$c=-1/\phi_0,\quad \alpha=\phi_1,\quad \beta=-\phi_2.$$

It is tempting to go on and try to extend the series to the term in x^3, but life isn't long enough!

References

[1] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, (Narosa New Delhi 1988).

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