On Klein's quartic curve

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1 Introduction

Berndt and Zhang [2] have drawn attention to three formulae at the foot of page 300 in Ramanujan's second notebook [17]. Of these formulae, the first is classical, the second trivial, and the third original with Ramanujan. The authors preface a fairly lengthy modular-form proof with the hope that "more natural proofs" of these entries might be found. This work appears also in [3] (p. 305–313, p. 466–473) and [4] (p. 176–185), along with treatments of related identities from page 239 of the notebook.

The triad of modular forms that these equations serve to connect goes back to Klein (see Conclusion). They parametrise the plane quartic curve given by equation (1a) below. Known properties of theta-functions imply linear transformations of the plane that leave the curve invariant (cf. [6]), and have the further interest that they comprise a simple group of order 168.

Recently Liu [16] (Theorem 6) has presented further relations among the forms (strictly, their images under modular inversion). In this paper I first derive the page 300 formulae from known modular-form results with especially short, elegant proofs, by series and product manipulations, that can be quoted from the literature. Many additional formulae, including some new ones, then follow by elementary ("high-school") algebra.

A trace of mystery does remain, for Ramanujan's third entry is not in its simplest form. To give a complete result, it must be combined with another entry in [17], namely 5(i) on page 254, which answers to equation (1.16) in [16] and to my Corollary 1. Here, for want of a tactical proof, I follow Hecke [9]. However, being essentially a fortified heuristic argument, this approach might perhaps revive something in the original line of thought.

2 The main results

We shall need [10] the product form

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}z)(1+q^{2n-1}z^{-1})$$

for the theta series

$$\theta(z,q)=\sum_{n=-\infty}^{\infty}z^n q^{[n]},$$

where $|q|<1$ and $[n]$ in the exponent means n^2. Write $q=\exp(\pi \mathrm{i}\omega)$ and $x=q^2$. The eta-function is defined then by

$$q^{-1/12}\eta(\omega)=\theta(-q,q^3)=\prod_{n=1}^{\infty}(1-q^{2n})=\prod_{n=1}^{\infty}(1-x^n).$$

Our initial concern is the last four lines on page 300 of [17]. To translate these results to present notation, let

$$u=q^{1/28}\theta(-q,q^7),\quad v=q^{9/28}\theta(-q^3,q^7),\quad w=q^{25/28}\theta(-q^5,q^7),$$

(these being $\sum_{n=-\infty}^{\infty}(-1)^n q^{[14n+j]/28}$ for $j = 1, 3, 5$) and write

$$h = \eta(\omega), \quad H = \eta(7\omega).$$

Finally, let

$$\Omega(\omega) = h^7/H + 13(hH)^3 + 49H^7/h.$$

Proposition 1 *We have*

$$u^2/v - v^2/w + w^2/u = 0, \tag{1a}$$

$$uvw = hH^2, \tag{1b}$$

$$v/u^2 - w/v^2 + u/w^2 = [\Omega(\omega)]^{1/3}/H^3. \tag{1c}$$

3 Start of the proofs

Equation (1b) is an immediate consequence of the product formulae. For symmetry and consistency with [7], I make a change of notation

$$(b, c, d) = (u, -w, v).$$

In this context the sign $\sum$ will denote summation over a *cyclic* permutation of b, c, d.

Lemma 1 *We have*

$$1 + \eta(\omega/7)/\eta(7\omega) = -\sum(c/b). \tag{2}$$

Proof. Equation (2) specialises [1, Lemma 6] to "$q = 7$". □

Remark The formula occurs as entry 17(v) of [17] (p. 239). The proof in [1] uses Watson's quintuple product identity, for which the authors give a non-elementary proof. See [12] for a simple proof, based on the fact that every integer $m + n\sqrt{-2}$ is congruent to 0, 1 or -1 (mod $1 + \sqrt{-2}$). For history, see [19].

Lemma 2 *Equation (1a) holds.*

Proof. By equation (2)

$$-[\eta(\omega/7)/\eta(7\omega)]^3 = 7 + 3\sum[2b/c + c/b + (c/b)^2 + b^2/cd + cd/b^2] + \sum(c/b)^3$$

The right-hand side may be brought to the form $A_0 + A_1 y + A_2 y^2 + \ldots + A_6 y^6$, where $y = x^{1/7}$ and each A_j is a Laurent series in x. Noting that $by^{-1/8}, cy^{-25/8}$ and $dy^{-9/8}$ are such series, we see that so too are $(b/c)y^3, (c/d)y^{-2}$ and $(d/b)y^{-1}$. The terms are easily sorted, to produce the formula $A_3 y^3 = c/b + (d/c)^2 + b^2/cd$.

However, in the Jacobian identity [11]

$$\prod_{n=1}^{\infty}(1 - q^{2n})^3 = (1/2)\sum_{-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)}$$

the series is lacunary; we cannot have $n(n+1) \equiv 4 \pmod 7$, i.e. $(2n+1)^2 \equiv 3 \pmod 7$, because 3 is not a quadratic residue. Thus $A_3 = 0$, giving

$$d^2/c + b^2/d + c^2/b = 0,$$

equivalent to equation (1a). □

For the next results, write $T = (h/H)^4$. (The function T is well known as a "Hauptmodul" for the principal congruence subgroup $\Gamma_0(7)$.)

Lemma 3 *We have both*

$$8 + T = -\sum(c^2 d/b^3), \quad 5 + T = \sum(b^3/c^2 d).$$

Proof. These relations are established in [15] (Eq. (5.20)-(5.21)) and also [18] (p. 242-243). □

Remark The formulae appear on page 300 of [17], separated by a horizontal dash from the last four lines. But the meanings of u, v, w differ. In terms of those below, the ones above are respectively $v/w, u/v$ and w/u.

4 Algebraic interlude

I digress briefly, reinterpreting b, c, d merely as indeterminates, and introducing five cyclic-permutation invariants as follows.

$$s = bcd, \quad p = \sum b^3c, \quad q = \sum b^2c^3, \quad r = \sum bc^5, \quad t = \sum b^7.$$

These particular expressions arise in the expansion

$$\prod(a + b\tau^3 + c\tau^5 + d\tau^6) = a^7 + 14sa^4 - 7pa^3 + 14qa^2 - 7(r + s^2)a + (t + 7sp),$$

where the product is over all seventh roots of unity τ. (Of course, the new q is not the same as $q = \exp(\pi i\omega)$, both above, and defining $\Theta(\omega)$ below.)

There are five of these quantities, dependent on only three indeterminates, so we expect two independent algebraic relations connecting them. To find such relations, begin computing products $p^2, pq, q^2, pr, ...$ in order of increasing degree. Note that, by cyclic invariance, $p \sum b^2c^3 = \sum pb^2c^3$. Thus we find

$$pq = \sum b^5c^4 + rs + 3s^3, \tag{3a}$$

$$q^2 = \sum b^4c^6 + 2ps^2, \tag{3b}$$

$$pr = \sum b^4c^6 + ps^2 + ts, \tag{3c}$$

$$p^3 = \sum b^9c^3 + 3pqs - 3s^4, \tag{3d}$$

$$qt = \sum b^9c^3 + \sum b^2c^{10} + rs^2, \tag{3e}$$

$$r^2 = \sum b^2c^{10} + 2s \sum b^5c^4. \tag{3f}$$

Subtracting equation (3c) from equation (3b) gives a first relation

$$q^2 + st = p(r + s^2). \tag{4}$$

For the second relation, eliminate $\sum b^2c^{10}$ between equations (3e) and (3f). After use of equations (3a) and (3d) to eliminate $\sum b^5c^4$ and $\sum b^9c^3$, we arrive at

$$qt + 5pqs = p^3 + r^2 + 3rs^2 + 9s^4. \tag{5}$$

5 The proof concluded

Now let b, c, d revert to their original meanings. Lemma 2 gives $p = 0$, so equations (4) and (5) reduce to

$$q^2 + st = 0, \quad qt = r^2 + 3rs^2 + 9s^4. \tag{6}$$

Thence, by eliminating t we obtain the formula

$$q^3 = -s(r^2 + 3rs^2 + 9s^4). \tag{7}$$

The second relation in Lemma 3 reads

$$T = r/s^2 - 5. \tag{8}$$

The first one reduces also to (8), by use of equation (3a) with $p = 0$.

Lemma 4 *We have* $[\sum(b/c^2)]^3 = \Omega/H^9$.

Proof. Equations (7)-(8) give $q^3 = -s^5(49 + 13T + T^2)$. But $s = -hH^2$ by equation (1b), so $(q/s^2)^3 = \Omega/H^9$ as claimed. □

To simplify (1c), write $Q(m, n) = m^2 + mn + 2n^2$, a binary quadratic form of discriminant (-7), and let $\Theta(\omega) = \sum_{n,m=-\infty}^{\infty} q^{Q(m,n)}$. In [17] (p. 254), $\Theta(\omega)$ appears as its well known Lambert series.

Corollary 1 *We have* $\Omega(\omega) = \Theta^3(\omega)$ *so* $v/u^2 - w/v^2 + u/w^2 = \Theta/H^3$.

Proof. (Sketch). As theta-series for the sum of three copies of Q, Θ^3 is a modular form of weight 3, and of "Nebentypus", for $\Gamma_0(7)$. So too is $E = h^7/H$ by [9] (p. 936-937) (see also [7]). It follows that Θ^3/E is a modular function, and hence a rational function of the Hauptmodul T. Moreover, $\Omega/E = (13 + T + 49/T)/T$ is manifestly of that form.

Now, replacement of ω by $-1/7\omega$ interchanges T with $49/T$ and leaves $\Theta^3(\omega)/\Omega(\omega)$ invariant. Thus the Laurent series in $1/T$ for $T\Theta^3/E$ must be a linear combination of terms of the form $T^i + (49/T)^i$. The q-series expansion then determines the coefficients. □

6 The T-formulae

Lemma 3 gives expressions by T for two cyclic sums of the form

$$\sum b^\beta c^\gamma d^\delta, \quad \beta + \gamma + \delta = 0.$$

As next illustrated, formulae of this pattern extend indefinitely.

Proposition 2 *We have*

$$\sum (c^2 d/b^3) = -(8 + T) \tag{9a}$$

$$\sum (b^3/c^2 d) = 5 + T \tag{9b}$$

$$\sum (cd^4/b^5) = -(46 + 13T + T^2) \tag{9c}$$

$$\sum (b^5/cd^4) = 3 \tag{9d}$$

$$\sum (c^4 d^2/b^6) = 54 + 14T + T^2 \tag{9e}$$

$$\sum (b^6/c^4 d^2) = 41 + 12T + T^2 \tag{9f}$$

$$\sum (c^7/b^7) = -(57 + 14T + T^2) \tag{9g}$$

$$\sum (b^7/c^7) = -(289 + 126T + 19T^2 + T^3) \tag{9h}$$

$$\sum (b^8/c^3 d^5) = -(44 + 12T + T^2) \tag{9i}$$

$$\sum (b^9/c^6 d^3) = 248 + 114T + 18T^2 + T^3 \tag{9j}$$

$$\sum (b^{11}/c^5 d^6) = -(204 + 102T + 17T^2 + T^3) \tag{9k}$$

$$\sum (b^{14}/c^7 d^7) = -(1069 + 854T + 207T^2 + 23T^3 + T^4) \tag{9l}$$

Proof. Equations (3a)–(3f), expressing products of powers of p, q, r, t by cyclic sums, can be continued to products of arbitrarily high homogeneous degree in b, c, d. Let $p = 0$ in these relations and solve for the sums $\sum b^\beta c^\gamma$. When $\beta + \gamma$ is divisible by 3, the expressions of interest prove to be functions of r and s alone. Thus

$$\begin{aligned}
\sum b^5c^4 &= -s(r + 3s^2)\\
\sum b^2c^{10} &= r^2 + 2rs^2 + 6s^4\\
\sum b^9c^3 &= 3s^4\\
\sum b^6c^9 &= -s(r^2 + 3rs^2 + 6s^4)\\
\sum b^{13}c^2 &= -s(r^2 + 2rs^2 + 9s^4)\\
\sum b^3c^{15} &= r^3 + 3r^2s^2 + 9rs^4 + 3s^6\\
\sum b^{10}c^8 &= s^2(r^2 + 4rs^2 + 9s^4)\\
\sum b^{17}c &= -(r^3 + 2r^2s^2 + 7rs^4 - 6s^6)\\
\sum b^{21} &= -(r^4 + 3r^3s^2 + 12r^2s^4 + 9rs^6 + 24s^8)/s\\
\sum b^7c^{14} &= -s(r^3 + 4r^2s^2 + 11rs^4 + 9s^6)\\
\sum b^{14}c^7 &= -s^3(r^2 + 4rs^2 + 12s^4).
\end{aligned}$$

The results then follow by equation (8). □

Remark Equations (9a)–(9h) form four pairs under negation of β, γ, δ, whereas (9i)–(9l) all have $\beta > 0$. The companion for (9i) involves the sum $\sum(c^3d^5/b^8)$, i.e. $\sum(c^{11}d^{13})/s^8$, where the numerator has homogeneous degree 24. It failed to appear only because we did not proceed so far.

Equations (1.21)–(1.27) of [17] correspond respectively to my equations (1a), (9a), (9b), (9g), (9h), (9c) and (1c). Equation (1.19) answers to (2). Further, (1.28) comes easily by combining equation (7) with the formula $\sum b^7c^7 = -qs(r + 2s^2)$, which arises from developing the degree-14 products t^2, pq^2 and p^2r. Finally, (1.20) reduces to a combination of (2) with the relation $p\sum b^3c^2 = 0$.

7 Conclusion

Equations (9d)–(9f) and (9i)–(9l), while possibly new, are seen to be only part of a longer sequence of such formulae. Every product $p^Xq^Yt^Z$ of degree $4X + 5Y + 7Z \equiv 0 \pmod 3$ has the form $(pq)^X(qt)^Zq^{3K}$ so, by equations (6)–(7), becomes 0 (if $X > 0$) or a rational function of r and s after we put $p = 0$.

For prior occurrence of equation (1a), see equations (4) and (7) in [8] (Bd II, p. 393–394), and equation (8) of [8] (Bd I, p. 730). The parametrisation in [14] (p. 456) evidently results from a 7-partition of $\theta(-q^7, q^3)$. Burnside [5] (p. 364, p. 497) used Klein's quartic curve for illustration, while Hurwitz [13], whose method I followed in [6], solved (1a) as a Diophantine equation.

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28+17. For x^2 read x_2.
28+22. Immediately following the equation insert: "For this quadratic in V to have integer solutions, the discriminant D must be a perfect square. However, with $X = WU$ we have $D = (1-X)^2(1+4X)$." (The final copy avoided use of X, after the referee noted its possible confusion with x^5.)
29+9. The left side of equation (5) should be $8VW^2$. Several subsequent occurrences of v then become vw^2.
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