A random hopscotch problem

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1 Introduction

Chapter 4 of [1], written by David Berengut, is devoted to the following "random hopscotch" problem. A counter moves along $\mathbb{Z}^+$, starting at 0, according to rolls of an unbiased N–faced die. What is the probability p_n that the counter lands at n? The author considers the case $N = 3$, and says "there is no easy way of deriving an explicit formula for p_n from the recursive relation." Well, of course there is, and that is the method of generating functions.

2 The solution

Let p_n be the probability that n is "encountered" (pun intended!).

It is easy to show that for $n = 1, \ldots, N$ if we set $p_0 = 1$,

$$p_n = \frac{1}{N}\left(p_{n-1} + \cdots + p_0\right)$$

and that for $n > N$,

$$p_n = \frac{1}{N}\left(p_{n-1} + \cdots + p_{n-N}\right).$$

Thus we find that the generating function is given by

$$P(t) = \sum_{n\geq 0} p_n t^n = \frac{N}{N - (t + t^2 + \cdots + t^N)}.$$

By a theorem of Abel,

$$\lim_{n\to\infty} p_n = \lim_{t\to 1}(1-t)P(t) = \frac{2}{N+1}.$$

This is not surprising, since the average roll is $\frac{N+1}{2}$. Indeed, if we use partial fractions, we find

$$P(t) = \frac{2}{N+1}\cdot\frac{1}{1-t} + \frac{N(N-1) + (N-1)(N-2)t + \cdots + 2\times 1t^{N-2}}{N + (N-1)t + \cdots + 1t^{N-1}}.$$

It follows that

$$p_n = \frac{2}{N+1} + A_1\alpha_1^n + \cdots + A_{N-1}\alpha_{N-1}^n,$$

where $\alpha_1, \cdots, \alpha_{N-1}$ are the zeroes of

$$Nz^{N-1} + (N-1)z^{N-2} + \cdots + 1 = 0$$

and $A_1, \cdots, A_{N-1}$ are certain constants. It is not hard to show that the α_k are all of modulus less than 1, and so $p_n \to \frac{2}{N+1}$ as $n \to \infty$. In the case $N = 2$, $p_n = \frac{2}{3} + \frac{1}{3}(-\frac{1}{2})^n$. In the case $N = 3$, $p_n = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)^n \cos n\theta$, where $\cos\theta = -\frac{1}{\sqrt{3}}$. In

the case $N = 4$, $p_n = \dfrac{2}{5} + \dfrac{1}{5}(-\alpha)^n + \dfrac{2}{5}\beta^n \cos n\phi$
where $\alpha = \dfrac{3 + \sqrt[3]{60\sqrt{6} + 135} - \sqrt[3]{60\sqrt{6} - 135}}{12} \approx 0.6058295861$,
$\beta = \dfrac{1}{2\sqrt{\alpha}} \approx 0.6423840736$, and $\cos\phi = \sqrt{\alpha}\left(\alpha - \dfrac{3}{4}\right)$.

References

[1] D. Berengut, in *The Mathematical Gardner*, ed. D. A. Klarner, (Wadsworth International 1981), 51–59.

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