On Weighted Means preserving Regular Variation

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Abstract

We give a necessary and sufficient condition for the weight sequence such that the classical weighted means of arbitrary order preserves regular variation of the target sequence. Some other properties of this class of weights are also considered.

Key words: Weighted Means, Regular Variation, Karamata.
MSC: Primary 26A12.

1 Introduction

Denote by $M_n^{(r)}(p;c)$ the classical weighted mean of order r with positive weights $p=\{p_i\}_{i=1}^n$ and the target sequence $c=\{c_i\}_{i=1}^n$ (Definition 2, below). In this article we shall solve the following problem:
Characterise somehow the weight sequence p such that the asymptotic equivalence

$$M_n^{(r)}(p;k) \sim k_n \qquad (n\to\infty), \tag{1}$$

takes place for each $r\in\mathbb{R}$ and each regularly varying sequence k of arbitrary index.

It follows from (1) that, for properly chosen weight sequence, the mean $M_n^{(r)}(p;k)$ varies regularly independently of order r.

Denoting $\sum_{i\le n} p_i := P_n$ and $\sum_{i\le n} P_i := Q_n$, it turns out that the condition

$$n(1-Q_{n-1}/Q_n)\to\infty \qquad (n\to\infty), \tag{2}$$

is necessary and sufficient for (1) to hold.

More explicitly, it is enough to take P_n in the form

$$P_n=\exp(a_n+\sum_{m\le n} b_m/m),$$

where $\{a_n\}$ is a non-decreasing sequence and $\{b_n\}$ is an unbounded sequence of positive numbers.

2 Preliminaries

We begin with some basic definitions from Karamata's theory of regularly varying sequences

Definition 1 It is said that a sequence $\{k_n\}_{n\ge1}$ of positive numbers is regularly varying with index a if it can be represented in the form

$$k_n := n^a\ell_n,\quad a\in\mathbb{R},\quad n\in\mathbb{N};$$

where $\{\ell_n\}_{n\geq 1}$ is a *slowly varying sequence* (SVS) i.e. satisfying

$$\ell_{[\lambda n]} \sim \ell_n \quad (n \to \infty),$$

for each positive λ(cf. [1, 2]).

Some examples of slowly varying sequences are

$$\log^b n,\ \ b \in \mathbb{R};\ \log^c(\log n),\ \ c \in \mathbb{R};\ \exp(\log^d n),\ 0 < d < 1;\ \exp(\log n/\log\log n);\ n > n_0.$$

For $n \leq n_0$ we can put $\ell_n = 1$.

Note that, if $\{c_n\}$ and $\{d_n\}$ are SVS, then also

$$\{c_n + d_n\};\ \ \{c_n \cdot d_n\};\ \ \ \{c_n/d_n\},$$

are SVS.

In the sequel we shall need the following lemma [2].

Lemma 1 *We have*

$$\sup_{m\leq n}(m\ell_m) \sim n\ell_n;\ \ \inf_{m\leq n}(\ell_m/m) \sim \ell_n/n \quad (n \to \infty).$$

Recall the definition of weighted means $M_n^{(r)}(p;c)$ [3, p. 72].

Definition 2 For a sequence of positive numbers $c = \{c_i\}_{i=1}^n$ and positive weights $p = \{p_i\}_{i=1}^n$, the weighted mean of order r is defined as

$$M_n^{(r)}(p;c) = \left(\frac{\sum_{i=1}^n p_i c_i^r}{P_n}\right)^{1/r}, \quad r \in \mathbb{R}/\{0\};$$

$$M_n^{(0)}(p;c) = \left(\prod_{i=1}^n c_i^{p_i}\right)^{1/P_n},$$

where $P_n := \sum_{i=1}^n p_i$.

The following lemma will be useful [3, p. 76].

Lemma 2 *The weighted mean $M_n^{(r)}(p;c)$ is an increasing function of r.*

3 Results

Denote by P the class of all sequences p of positive numbers satisfying the condition (2). The manner in which the class P acts on the class of regularly varying sequences is given in the next theorem.

Theorem 1 *For each $r \in \mathbb{R}$ and every regularly varying sequence $k = \{k_m\}_{m\geq 1}$ of arbitrary index, as $n \to \infty$, the following are equivalent*

$$i)\ \ Q_n/nP_n \to 0;\ \ \ ii)\ \ \sum_{m\leq n} k_m p_m \sim k_n P_n;$$

$$iii)\ \ M_n^{(r)}(p;k) \sim k_n;\ \ \ iv)\ \ \frac{\sum_{m\leq n} p_m \log k_m}{P_n} - \log k_n \to 0,$$

where $P_n := \sum_{m\leq n} p_m$ and $Q_n := \sum_{m\leq n} P_m$.

Note that $i)$ is equivalent to the condition (2) above.

If $p \in P$ then $\{P_n\}$ and $\{Q_n\}$ are of rapid growth in the sense that $n^\alpha/P_n \to 0 \quad (n \to \infty)$ for each $\alpha \in \mathbb{R}$. What is the representation form of P_n is an open question. We are satisfied here with the following

Theorem 2 *Let $\{a_n\}_{n\geq 1}$ be an arbitrary non-decreasing sequence and $\{b_n\}_{n\geq 1}$ an arbitrary unbounded sequence of positive numbers. Then*

$$P_n = \exp(a_n + \sum_{m\leq n} b_m/m), \tag{3}$$

implies $p \in P$.

4 Proofs

Proof of Theorem 1

For $k_n = n$, by Abel's partial summation we get

$$\sum_{m\leq n} mp_m = (n+1)P_n - Q_n.$$

Hence, as $n \to \infty$, $\sum_{m\leq n} mp_m \sim nP_n$ is equivalent to $Q_n/nP_n \to 0$, i.e. the condition $i)$ is necessary for $ii)$ to hold. To prove that it is also sufficient, the following lemmas are needed.

Lemma 3 *Denote, as usual, $\Delta r_n := r_{n+1} - r_n$. If, as $n \to \infty$, $r_n \to \infty$ and $\frac{\Delta t_n}{\Delta r_n} \to s$, then also $\frac{t_n}{r_n} \to s$.*

This is a well-known classical assertion (cf. [4, p. 30]).

Lemma 4 *Under the condition $i)$ of Theorem A, for each real α we have*

$$i)\ n^\alpha Q_n \to \infty; \quad ii)\ \sum_{m\leq n} m^\alpha p_m \sim n^\alpha P_n \qquad (n \to \infty).$$

Proof. Since $nP_n/Q_n \to \infty$, for $n > n_0$ and fixed $\alpha \in \mathbb{R}$, we have

$$\frac{nP_n}{Q_n} > |\alpha| + 1 \text{ i.e. } \frac{Q_n - Q_{n-1}}{Q_n} > \frac{|\alpha|+1}{n} \text{ i.e. } \frac{Q_{n-1}}{Q_n} < 1 - \frac{|\alpha|+1}{n} < \exp(-\frac{|\alpha|+1}{n}).$$

It follows

$$Q_n \gg \exp((|\alpha|+1)\sum_{m\leq n} 1/m) \gg \exp((|\alpha|+1)\log n).$$

Hence

$$n^\alpha Q_n \gg n^{\alpha+|\alpha|+1} \qquad (n \to \infty),$$

i.e. the part $i)$ is proved.

By (2) we get

$$\frac{P_n \Delta n^\alpha}{\Delta(n^{\alpha-1}Q_{n-1})} = \frac{P_n \Delta n^\alpha}{n^{\alpha-1}P_n + Q_n \Delta n^{\alpha-1}} \to \alpha \qquad (n \to \infty).$$

Hence, applying part $i)$, Lemma 3 and (2), for each real α we obtain

$$S_n := \sum_{m\leq n} P_m \Delta m^\alpha \sim \alpha n^{\alpha-1} Q_n = o(n^\alpha P_n) \quad (n \to \infty).$$

Therefore, by partial summation we have

$$\sum_{m\le n} m^\alpha p_m = (n+1)^\alpha P_n - S_n = (n+1)^\alpha P_n + o(n^\alpha P_n) \quad (n\to\infty),$$

and the part $ii)$ of Lemma 4 is also proved.

Now, for $k_n := n^\alpha \ell_n,\ \alpha\in\mathbb{R}$, using lemmas 1 and 4 (part $ii)$), we get

$$\sum_{m\le n} m^\alpha \ell_m p_m \le \sup_{m\le n}(m\ell_m)\sum_{m\le n} m^{\alpha-1}p_m \sim n^\alpha \ell_n P_n \quad (n\to\infty),$$

and

$$\sum_{m\le n} m^\alpha \ell_m p_m \ge \inf_{m\le n}(\ell_m/m)\sum_{m\le n} m^{\alpha+1}p_m \sim n^\alpha \ell_n P_n \quad (n\to\infty).$$

Hence

$$1\le \liminf_n \frac{\sum_{m\le n} m^\alpha \ell_m p_m}{n^\alpha \ell_n P_n} \le \limsup_n \frac{\sum_{m\le n} m^\alpha \ell_m p_m}{n^\alpha \ell_n P_n} \le 1,$$

i.e., parts $i)$ and $ii)$ of Theorem A are equivalent.

Remark Note that the previous proof yields the following:

$$i)\ \{p_n\}_{n\ge1}\in P; \quad ii)\ \{k_n p_n\}_{n\ge 1}\in P,$$

are equivalent.

Now we shall prove $ii) \iff iii)$. Indeed, putting $r=1$ in $iii)$ it follows

$$M_n^{(1)}(p,k) = \sum_{m\le n} k_m p_m/P_n \sim k_n \quad (n\to\infty).$$

Hence $iii) \Rightarrow ii)$.

Assume now that $ii)$ holds. From Definition 1, it follows that if k_n varies regularly with index α, then k_n^r also varies regularly with index αr.

Therefore, by $ii)$

$$\sum_{m\le n} k_m^r p_m \sim k_n^r P_n \quad (n\to\infty),$$

and, by Definition 2,

$$M_n^{(r)}(p;k)\sim (k_n^r)^{1/r}\sim k_n \quad \forall r\in\mathbb{R}/\{0\}.$$

In the case $r=0$, applying Lemma 2, we obtain

$$k_n \sim M_n^{(-1)}(p;k) < M_n^{(0)}(p;k) < M_n^{(1)}(p;k)\sim k_n \quad (n\to\infty).$$

Hence

$$M_n^{(0)}(p;k)\sim k_n \qquad (n\to\infty), \tag{4}$$

and the proof is done. □

Taking logarithm on both sides of (4) we obtain the assertion $iv)$ of Theorem A. Therefore $i)$ is sufficient for $iv)$ to hold. To prove that it is also necessary, put $k_n = n$ in $iv)$; then, by partial summation we get

$$\sum_{m\le n} P_m \Delta \log m/P_n = \log(n+1) - \sum_{m\le n} p_m \log m/P_n = \log(n+1) - \log n + o(1) = o(1).$$

Since, for $m \in [1, n]$

$$\Delta \log m = \log(1 + \frac{1}{m}) > \frac{2}{2m+1} > \frac{1}{2m} \geq \frac{1}{2n},$$

we obtain

$$0 < Q_n/nP_n < 2 \sum_{m \leq n} P_m \Delta \log m / P_n \to 0 \quad (n \to \infty).$$

i.e. $p \in P$.

Proof of Theorem 2

Proof. For $P_n = \exp(a_n + \sum_{m \leq n} b_m/m)$, we get

$$\frac{\Delta(nP_n)}{\Delta Q_n} = n(1 - \frac{P_{n-1}}{P_n}) + \frac{P_{n-1}}{P_n} > n(1 - \exp(-\Delta a_{n-1} - b_n/n)) \geq n(1 - \exp(-b_n/n)) \to \infty,$$

since $b_n \to \infty \quad (n \to \infty)$.

Hence, by Lemma 3, we obtain $nP_n/Q_n \to \infty \quad (n \to \infty)$ i.e. $p \in P$. □

Note that if $a_n \to a,\ b_n \to b,\ a, b \in \mathbb{R}$, then (3) becomes the well-known representation form for a *regularly varying* sequence of index b [1, p. 52].

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