Euler Polynomials of Higher Order Involving the Stirling Numbers of the Second Kind

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Abstract

The object of the present note is to prove certain new explicit formulae for the Euler numbers and polynomials of higher order, these representations include the Stirling numbers of the second kind.

Key words: Euler numbers of higher order; Euler polynomials of higher order; Stirling numbers of the second kind; difference operator.
MSC: Primary 11B68, 33E20; Secondary 11B73.

1 Introduction

In the usual notations, let $E_n^{(\alpha)}(x)$ denote the Euler polynomial of higher order of degree n in x, defined by (see [1, P. 66, Eq. (64)])

$$\left(\frac{2}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!}, \quad (|z|<\pi) \tag{1}$$

for an arbitrary (real or complex) parameter α.

From the generating relation (1), it is fairly straightforward to deduce the following formulae

$$E_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{E_k^{(\alpha)}}{2^k}\left(x-\frac{\alpha}{2}\right)^{n-k}$$

$$E_n^{(\alpha)}(x) = (-1)^n E_n^{(\alpha)}(\alpha - x)$$

in terms of the Euler numbers of higher order $E_n^{(\alpha)}$ (see [1, P. 66, Eq. (65)]).

Recently, Qiu-Ming Luo *et al* gave certain new classes of recursion formulae for the Euler numbers and polynomials of higher order (see [2, p. 2, Eqs. (9) and (10); p. 3, Eqs. (24) and (25)]). In the present note we first prove the following explicit formula for the Euler polynomials of higher order

$$E_n^{(\alpha)}(x) = \sum_{s=0}^{n}\binom{n}{s}x^{n-s}\sum_{k=0}^{s}\frac{(-1)^k k!}{2^k}\binom{\alpha+k-1}{k}S(s,k) \tag{2}$$

We shall also apply the representation (2) in order to derive certain interesting special cases.

2 Proof of the Explicit Formula (2)

By Taylor's expansion and Leibniz's rule, the generating relation (1) yields

$$E_n^{(\alpha)}(x) = \sum_{s=0}^{n}\binom{n}{s}x^{n-s}D_z^s\left\{\left(\frac{2}{e^z+1}\right)^{\alpha}\right\}\Big|_{z=0}, \quad D_z=\frac{d}{dz}. \tag{3}$$

Since

$$(1+w)^{-\alpha} = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} (-w)^k \quad (|w|<1),$$

setting $1+w = (e^z+1)/2$, and applying the binomial theorem, we find from (3) that

$$E_n^{(\alpha)}(x) = \sum_{s=0}^{n} \binom{n}{s} x^{n-s} \sum_{k=0}^{s} \frac{(-1)^k}{2^k} \binom{\alpha+k-1}{k} D_z^s\{(e^z-1)^k\}|_{z=0} \tag{4}$$

Now make use of the well-know formula (cf. [3, 24.1.4, I. def. B and C])

$$\begin{aligned} (e^z-1)^k &= \sum_{r=k}^{\infty} \frac{z^r}{r!} \Delta^k 0^r \\ S(r,k) &= \frac{1}{k!} \Delta^k 0^r, \end{aligned} \tag{5}$$

where $S(r,k)$ denotes the Stirling numbers of the second kind, defined by

$$x^r = \sum_{k=0}^{r} \binom{x}{k} k! S(r,k),$$

where, for convenience,

$$\Delta^k a^r = \Delta^k x^r|_{x=a} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (a+j)^r, \tag{6}$$

Δ being the difference operator defined by (cf. [3, p.822,III])

$$\Delta f(x) = f(x+1) - f(x),$$

so that, in general (cf. [3, p.823,24.1.1,C]),

$$\Delta^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+j),$$

by formula (5) readily yields

$$D_z^s\{(e^z-1)^k\}|_{z=0} = \Delta^k 0^s = k! S(s,k) \tag{7}$$

and upon substituting this value in (4), leads us immediately to the explicit formula (2).

3 Application

The first special case of our formula (2) when $x = \frac{\alpha}{2}$, $E_n^{(\alpha)} = 2^n E_n^{(\alpha)}\left(\frac{\alpha}{2}\right)$, gives us the following for the Euler numbers of higher order $E_n^{(\alpha)}$ can be written as (new formula)

$$E_n^{(\alpha)} = \sum_{s=0}^{n} \binom{n}{s} 2^s \alpha^{n-s} \sum_{k=0}^{s} \frac{(-1)^k k!}{2^k} \binom{\alpha+k-1}{k} S(s,k),$$

further, upon setting $\alpha = 1$, we provide an interesting result for the classical Euler numbers $E_n :\equiv E_n^{(1)}$

$$E_n = \sum_{s=0}^{n} \binom{n}{s} 2^s \sum_{k=0}^{s} \frac{(-1)^k k!}{2^k} S(s,k).$$

The second special case of our formula (2), for $\alpha = 1$, obtain to an explicit formula for the classical Euler polynomials $E_n(x)$

$$E_n(x) = \sum_{s=0}^{n} \binom{n}{s} x^{n-s} \sum_{k=0}^{s} \frac{(-1)^k k!}{2^k} S(s,k)$$

This last representation, for $x = 0$ in (2) yields

$$E_n^{(\alpha)}(0) = \sum_{k=0}^{n} \frac{(-1)^k k!}{2^k} \binom{\alpha + k - 1}{k} S(n,k),$$

further, upon setting $\alpha = 1$, we have

$$E_n(0) = \sum_{k=0}^{n} \frac{(-1)^k k!}{2^k} S(n,k).$$

4 Acknowledgment

The author was supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200).

References

[1] H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions* (Kluwer Academic Publishers Dordrecht 2001).

[2] Q.-M. Luo, Y.-M. Zheng and F. Qi, *Euler numbers of higher order and Euler polynomials of higher order*, Henan Sci. **21** (2003), 1–6 (in Chinese).

[3] M. Abramowitz and I.A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series 55, 4th printing (National Bureau of Standards Washington 1965).

[4] Q.-M. Luo, F. Qi, and L. Debnath, *Generalizations of Euler numbers and polynomials*, Int. J. Math. Math. Sci., **2003** (2003), 3893–3901.

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Received 2 December 2003, accepted 20 January 2004.