

Ramanujan's approximation to the zero of a continued fraction

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On page 48 of the Lost Notebook [2], Ramanujan writes

If

$$1 - \frac{aq}{1} - \frac{aq^2}{1} - \frac{aq^3}{1} - \dots = 0$$

then

$$q = \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} - \frac{1266}{a^8} + \frac{5289}{a^9} - \frac{22553}{a^{10}} + \dots$$

$$q = \frac{2}{a - 1 + \sqrt{(a+1)(a+5)}} + O\left(\frac{1}{a^8}\right)$$

$$\frac{2}{a - 1 + \sqrt{(a+1)(a+5)}} = \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} - \frac{1265}{a^8} + \frac{5275}{a^9} - \frac{22431}{a^{10}} + \dots$$

$$q = \frac{1}{\frac{a - 1 + \sqrt{(a+1)(a+5)}}{2} + \left(\frac{a + 3 - \sqrt{(a+1)(a+5)}}{a - 1 + \sqrt{(a+1)(a+5)}}\right)^3}. \quad (1)$$

He also writes

If

$$1 - \frac{q}{1} - \frac{q^2}{1} - \frac{q^3}{1} - \dots = 0$$

$$q = 0.5762$$

and later jottings include

$$\frac{1}{5\left(\frac{7}{9}\sqrt{3} - 1\right)} =$$

(Actually, in all the above, I have taken the liberty of replacing Ramanujan's x by the now more usual q .)

As we shall see, everything Ramanujan wrote is correct, except that the statement (1) should be

$$q = \frac{1}{\frac{a - 1 + \sqrt{(a+1)(a+5)}}{2} + \left(\frac{a + 3 - \sqrt{(a+1)(a+5)}}{a - 1 + \sqrt{(a+1)(a+5)}}\right)^3} + O\left(\frac{1}{a^{11}}\right).$$

Note that if the result (1) were exact, then putting $a = 1$ would give that the zero of $1 - \frac{q}{1} - \frac{q^2}{1} - \frac{q^3}{1} - \dots$ is $\frac{1}{5\left(\frac{7}{9}\sqrt{3} - 1\right)} \approx 0.57611879$, whereas in fact the zero ≈ 0.57614877 .

Let me start out by saying that it is well-known that the continued fraction can be written

$$1 - \frac{aq}{1} - \frac{aq^2}{1} - \frac{aq^3}{1} - \dots = \sum_{n \geq 0} \frac{(-1)^n a^n q^{n^2}}{(q)_n} / \sum_{n \geq 0} \frac{(-1)^n a^n q^{n^2+n}}{(q)_n}$$

where

$$(q)_0 = 1, \quad (q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n) \text{ for } n \geq 1.$$

Indeed, this continued fraction is known as the Rogers–Ramanujan continued fraction, which is discussed in Hardy and Wright, Chapter 19.

Thus, the continued fraction is zero when

$$\sum_{n \geq 0} \frac{(-1)^n a^n q^{n^2}}{(q)_n} = 0. \tag{2}$$

If we consider a large, it is obvious from the first two terms of this equation,

$$1 - \frac{aq}{1 - q} = 0,$$

that

$$q = \frac{1}{a} + o\left(\frac{1}{a}\right) \text{ as } a \rightarrow \infty.$$

If we want the solution of (2) as a series in powers of $\frac{1}{a}$ up to, say, the term in $\frac{1}{a^{11}}$, we can ignore all the terms in the sum on the left of (2) beyond the term $-\frac{a^3 q^9}{(q)_3}$, since

$$\frac{a^4 q^{16}}{(q)_4} = O\left(\frac{1}{a^{12}}\right).$$

So we consider

$$1 - \frac{aq}{1 - q} + \frac{a^2 q^4}{(1 - q)(1 - q^2)} - \frac{a^3 q^9}{(1 - q)(1 - q^2)(1 - q^3)} = 0$$

or, recalling that $q = O\left(\frac{1}{a}\right)$,

$$\begin{aligned} &1 - aq(1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + q^7 + q^8 + q^9 + q^{10} + q^{11}) \\ &+ a^2 q^4(1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 4q^6 + 4q^7 + 5q^8 + 5q^9) \\ &- a^3 q^9(1 + q + 2q^2 + 3q^3 + 4q^4 + 5q^5) = 0 \pmod{(a^{-12})}. \end{aligned}$$

If we write

$$q = \frac{1}{a} + \frac{c_1}{a^2} + \frac{c_2}{a^3} + \frac{c_3}{a^4} + \frac{c_4}{a^5} + \frac{c_5}{a^6} + \frac{c_6}{a^7} + \frac{c_7}{a^8} + \frac{c_8}{a^9} + \frac{c_9}{a^{10}} + \frac{c_{10}}{a^{11}} + O\left(\frac{1}{a^{12}}\right)$$

and substitute, we find that

$$\frac{0}{a} + \frac{-c_1 - 1}{a^2} + \frac{-c_2 - 2c_1}{a^3} + \frac{-c_3 - 2c_2 - c_1^2 + c_1}{a^4} \cdots = 0 \pmod{(a^{-12})},$$

and if we equate all the numerators to 0, we find (I used MAPLE, and it took less than 2 seconds!)

$$\begin{aligned} c_1 &= -1, \quad c_2 = 2, \quad c_3 = -6, \quad c_4 = 21, \quad c_5 = -79, \quad c_6 = 311 \\ c_7 &= -1266, \quad c_8 = 5289, \quad c_9 = -22553, \quad c_{10} = 97763, \end{aligned}$$

so

$$q = \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} - \frac{1266}{a^8} + \frac{5289}{a^9} - \frac{22553}{a^{10}} + \frac{97763}{a^{11}} + O\left(\frac{1}{a^{12}}\right).$$

Thus, we have certainly verified Ramanujan's first statement.

It is easy using MAPLE to verify all Ramanujan's other statements. Thus

$$\frac{1}{\frac{a-1+\sqrt{(a+1)(a+5)}}{2} + \left(\frac{a+3-\sqrt{(a+1)(a+5)}}{a-1+\sqrt{(a+1)(a+5)}}\right)^3}$$

$$= \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} - \frac{1266}{a^8} + \frac{5289}{a^9} - \frac{22553}{a^{10}} + \frac{97760}{a^{11}} + O\left(\frac{1}{a^{12}}\right).$$

But the real question is, how did Ramanujan discover the “pseudo-rational” functions with the “almost exact” series expansions?

I am going to hazard the following guess: The series that matters here is

$$s = 1 - z + 2z^2 - 6z^3 + 21z^4 - 79z^5 + 311z^6 - 1266z^7 + 5289z^8 - 22553z^9 + \dots$$

The reciprocal of this series is

$$s^{-1} = 1 + z - z^2 + 3z^3 - 10z^4 + 36z^5 - 137z^6 + 544z^7 - 2231z^8 + 9378z^9 + \dots$$

On the other hand, Ramanujan may have had in his kit-bag the fact that

$$\sqrt{1+6z+5z^2} = 1+3z-2z^2+6z^3-20z^4+72z^5-274z^6+1086z^7-4438z^8+18570z^9+\dots$$

He may have spotted that

$$s^{-1} = \frac{1-z+\sqrt{1+6z+5z^2}}{2} + z^7 - 12z^8 + 93z^9 + \dots$$

$$= \frac{1-z+\sqrt{1+6z+5z^2}}{2} + z(z^2-4z^3+15z^4+\dots)^3$$

and so

$$s = \frac{1}{\frac{1-z+\sqrt{1+6z+5z^2}}{2} + z(z^2-4z^3+15z^4+\dots)^3}.$$

I do not put it past Ramanujan to see all this, and further that

$$1+3z-\sqrt{1+6z+5z^2} = 2z^2(1-3z+10z^2+\dots),$$

$$1-z+\sqrt{1+6z+5z^2} = 2(1+z-z^2+\dots)$$

and so

$$\frac{1+3z-\sqrt{1+6z+5z^2}}{1-z+\sqrt{1+6z+5z^2}} = z^2-4z^3+15z^4+\dots$$

The rest is easy.

I carried the calculations a little further, and found

$$\begin{aligned}
q = & \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} - \frac{1266}{a^8} + \frac{5289}{a^9} - \frac{22553}{a^{10}} + \frac{97763}{a^{11}} - \frac{429527}{a^{12}} \\
& + \frac{1908452}{a^{13}} - \frac{8560532}{a^{14}} + \frac{38713510}{a^{15}} - \frac{176318081}{a^{16}} + \frac{808018789}{a^{17}} - \frac{3723242051}{a^{18}} \\
& + \frac{17239848937}{a^{19}} - \frac{80174546765}{a^{20}} + \frac{374319144257}{a^{21}} - \frac{1753833845882}{a^{22}} + \frac{8243964424236}{a^{23}} \\
& - \frac{38865436663306}{a^{24}} + \frac{183723384886286}{a^{25}} - \frac{870653110451115}{a^{26}} + \frac{4135450843397738}{a^{27}} \\
& - \frac{19684477775566090}{a^{28}} + \frac{93882029721904349}{a^{29}} - \dots
\end{aligned}$$

Note added: Bruce Berndt points out that Ramanujan's claims are discussed in [1].

References

- [1] B.C. Berndt, S.-S. Huang, J. Sohn and S.H. Son, *Some theorems on the Rogers–Ramanujan continued fraction in Ramanujan's Lost Notebook*, Trans Amer. Math. Soc. **352** (2000), 2157–2177.
[2] S. Ramanujan, *The Lost Notebook and other unpublished papers*, Narosa New Delhi 1988.

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