Ramanujan's approximation to the zero of a continued fraction

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On page 48 of the Lost Notebook [2], Ramanujan writes
If

$$1-\frac{aq}{1-}\frac{aq^2}{1-}\frac{aq^3}{1-}\;\cdots=0$$

then

$$q=\frac{1}{a}-\frac{1}{a^2}+\frac{2}{a^3}-\frac{6}{a^4}+\frac{21}{a^5}-\frac{79}{a^6}+\frac{311}{a^7}-\frac{1266}{a^8}+\frac{5289}{a^9}-\frac{22553}{a^{10}}+\;\cdots$$

$$q=\frac{2}{a-1+\sqrt{(a+1)(a+5)}}+\mathrm{O}\left(\frac{1}{a^8}\right)$$

$$\frac{2}{a-1+\sqrt{(a+1)(a+5)}}=\frac{1}{a}-\frac{1}{a^2}+\frac{2}{a^3}-\frac{6}{a^4}+\frac{21}{a^5}-\frac{79}{a^6}+\frac{311}{a^7}-\frac{1265}{a^8}+\frac{5275}{a^9}-\frac{22431}{a^{10}}+\cdots$$

$$q=\frac{1}{\dfrac{a-1+\sqrt{(a+1)(a+5)}}{2}+\left(\dfrac{a+3-\sqrt{(a+1)(a+5)}}{a-1+\sqrt{(a+1)(a+5)}}\right)^3}. \tag{1}$$

He also writes
If

$$1-\frac{q}{1-}\frac{q^2}{1-}\frac{q^3}{1-}\;\cdots=0$$

$$q=0.5762$$

and later jottings include

$$\frac{1}{5\left(\frac{7}{9}\sqrt{3}-1\right)}=$$

(Actually, in all the above, I have taken the liberty of replacing Ramanujan's x by the now more usual q.)

As we shall see, everything Ramanujan wrote is correct, except that the statement (1) should be

$$q=\frac{1}{\dfrac{a-1+\sqrt{(a+1)(a+5)}}{2}+\left(\dfrac{a+3-\sqrt{(a+1)(a+5)}}{a-1+\sqrt{(a+1)(a+5)}}\right)^3}+\mathrm{O}\left(\frac{1}{a^{11}}\right).$$

Note that if the result (1) were exact, then putting $a=1$ would give that the zero of $1-\frac{q}{1-}\frac{q^2}{1-}\frac{q^3}{1-}\;\cdots$ is $\frac{1}{5\left(\frac{7}{9}\sqrt{3}-1\right)}\approx 0.57611879$, whereas in fact the zero ≈ 0.57614877.

Let me start out by saying that it is well–known that the continued fraction can be written

$$1-\frac{aq}{1-}\frac{aq^2}{1-}\frac{aq^3}{1-}\;\cdots=\sum_{n\ge0}\frac{(-1)^n a^n q^{n^2}}{(q)_n}\Big/\sum_{n\ge0}\frac{(-1)^n a^n q^{n^2+n}}{(q)_n}$$

where

$$(q)_0 = 1, \quad (q)_n = (1-q)(1-q^2) \ \cdots \ (1-q^n) \text{ for } n \geq 1.$$

Indeed, this continued fraction is known as the Rogers–Ramanujan continued fraction, which is discussed in Hardy and Wright, Chapter 19.

Thus, the continued fraction is zero when

$$\sum_{n\geq 0} \frac{(-1)^n a^n q^{n^2}}{(q)_n} = 0. \tag{2}$$

If we consider a large, it is obvious from the first two terms of this equation,

$$1 - \frac{aq}{1-q} = 0,$$

that

$$q = \frac{1}{a} + \mathrm{o}\left(\frac{1}{a}\right) \quad \text{as } a \to \infty.$$

If we want the solution of (2) as a series in powers of $\frac{1}{a}$ up to, say, the term in $\frac{1}{a^{11}}$, we can ignore all the terms in the sum on the left of (2) beyond the term $-\frac{a^3q^9}{(q)_3}$, since

$$\frac{a^4 q^{16}}{(q)_4} = \mathrm{O}\left(\frac{1}{a^{12}}\right).$$

So we consider

$$1 - \frac{aq}{1-q} + \frac{a^2q^4}{(1-q)(1-q^2)} - \frac{a^3q^9}{(1-q)(1-q^2)(1-q^3)} = 0$$

or, recalling that $q = \mathrm{O}\left(\frac{1}{a}\right)$,

$$\begin{aligned} 1 - &aq(1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10}+q^{11}) \\ &+ a^2q^4(1+q+2q^2+2q^3+3q^4+3q^5+4q^6+4q^7+5q^8+5q^9) \\ &- a^3q^9(1+q+2q^2+3q^3+4q^4+5q^5) = 0 \pmod{\left(a^{-12}\right)}. \end{aligned}$$

If we write

$$q = \frac{1}{a} + \frac{c_1}{a^2} + \frac{c_2}{a^3} + \frac{c_3}{a^4} + \frac{c_4}{a^5} + \frac{c_5}{a^6} + \frac{c_6}{a^7} + \frac{c_7}{a^8} + \frac{c_8}{a^9} + \frac{c_9}{a^{10}} + \frac{c_{10}}{a^{11}} + \mathrm{O}\left(\frac{1}{a^{12}}\right)$$

and substitute, we find that

$$\frac{0}{a} + \frac{-c_1-1}{a^2} + \frac{-c_2-2c_1}{a^3} + \frac{-c_3-2c_2-c_1^2+c_1}{a^4} \cdots = 0 \pmod{\left(a^{-12}\right)},$$

and if we equate all the numerators to 0, we find (I used MAPLE, and it took less than 2 seconds!)

$$\begin{gathered} c_1 = -1, \ c_2 = 2, \ c_3 = -6, \ c_4 = 21, \ c_5 = -79, \ c_6 = 311 \\ c_7 = -1266, \ c_8 = 5289, \ c_9 = -22553, \ c_{10} = 97763, \end{gathered}$$

so

$$q = \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} - \frac{1266}{a^8} + \frac{5289}{a^9} - \frac{22553}{a^{10}} + \frac{97763}{a^{11}} + \mathrm{O}\left(\frac{1}{a^{12}}\right).$$

Thus, we have certainly verified Ramanujan's first statement.

It is easy using MAPLE to verify all Ramanujan's other statements. Thus

$$\frac{1}{\dfrac{a-1+\sqrt{(a+1)(a+5)}}{2}+\left(\dfrac{a+3-\sqrt{(a+1)(a+5)}}{a-1+\sqrt{(a+1)(a+5)}}\right)^3}$$
$$=\frac{1}{a}-\frac{1}{a^2}+\frac{2}{a^3}-\frac{6}{a^4}+\frac{21}{a^5}-\frac{79}{a^6}+\frac{311}{a^7}-\frac{1266}{a^8}+\frac{5289}{a^9}-\frac{22553}{a^{10}}+\frac{97760}{a^{11}}+\mathrm{O}\left(\frac{1}{a^{12}}\right).$$

But the real question is, how did Ramanujan discover the "pseudo–rational" functions with the "almost exact" series expansions?
I am going to hazard the following guess: The series that matters here is

$$s=1-z+2z^2-6z^3+21z^4-79z^5+311z^6-1266z^7+5289z^8-22553z^9+\cdots.$$

The reciprocal of this series is

$$s^{-1}=1+z-z^2+3z^3-10z^4+36z^5-137z^6+544z^7-2231z^8+9378z^9+\cdots.$$

On the other hand, Ramanujan may have had in his kit–bag the fact that

$$\sqrt{1+6z+5z^2}=1+3z-2z^2+6z^3-20z^4+72z^5-274z^6+1086z^7-4438z^8+18570z^9+\cdots.$$

He may have spotted that

$$\begin{aligned} s^{-1}&=\frac{1-z+\sqrt{1+6z+5z^2}}{2}+z^7-12z^8+93z^9+\cdots\\ &=\frac{1-z+\sqrt{1+6z+5z^2}}{2}+z(z^2-4z^3+15z^4+\cdots)^3 \end{aligned}$$

and so

$$s=\frac{1}{\dfrac{1-z+\sqrt{1+6z+5z^2}}{2}+z(z^2-4z^3+15z^4+\cdots)^3}.$$

I do not put it past Ramanujan to see all this, and further that

$$1+3z-\sqrt{1+6z+5z^2}=2z^2(1-3z+10z^2+\cdots),$$

$$1-z+\sqrt{1+6z+5z^2}=2(1+z-z^2+\cdots)$$

and so

$$\frac{1+3z-\sqrt{1+6z+5z^2}}{1-z+\sqrt{1+6z+5z^2}}=z^2-4z^3+15z^4+\cdots.$$

The rest is easy.
I carried the calculations a little further, and found

$$q = \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} - \frac{1266}{a^8} + \frac{5289}{a^9} - \frac{22553}{a^{10}} + \frac{97763}{a^{11}} - \frac{429527}{a^{12}}$$
$$+ \frac{1908452}{a^{13}} - \frac{8560532}{a^{14}} + \frac{38713510}{a^{15}} - \frac{176318081}{a^{16}} + \frac{808018789}{a^{17}} - \frac{3723242051}{a^{18}}$$
$$+ \frac{17239848937}{a^{19}} - \frac{80174546765}{a^{20}} + \frac{374319144257}{a^{21}} - \frac{1753833845882}{a^{22}} + \frac{8243964424236}{a^{23}}$$
$$- \frac{38865436663306}{a^{24}} + \frac{183723384886286}{a^{25}} - \frac{870653110451115}{a^{26}} + \frac{4135450843397738}{a^{27}}$$
$$- \frac{19684477775566090}{a^{28}} + \frac{93882029721904349}{a^{29}} - \cdots .$$

Note added: Bruce Berndt points out that Ramanujan's claims are discussed in [1].

References

[1] B.C. Berndt, S.-S. Huang, J. Sohn and S.H. Son, *Some theorems on the Rogers–Ramanujan continued fraction in Ramanujan's Lost Notebook*, Trans Amer. Math. Soc. **352** (2000), 2157–2177.

[2] S. Ramanujan, *The Lost Notebook and other unpublished papers*, Narosa New Delhi 1988.

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