# Gravity matters

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## 1 Geometry

One equation says it all:

$$G_{ab} = 8\pi T_{ab}. \tag{1}$$

On the left we have mathematics, represented by the Einstein tensor $G_{ab} = \mathrm{Ric}_{ab} - \frac{1}{2}Rg_{ab}$ of the spacetime metric $g_{ab}$, where Ric is the Ricci curvature tensor. The metric $g_{ab}$ determines geodesics and dynamics in general; $G_{ab}$ controls one half of the spacetime curvature tensor directly, and the remaining pieces, the Weyl curvature, indirectly through an essentially hyperbolic evolution.

On the right we have physics, represented by the stress-energy tensor $T_{ab}$ built from the matter fields. Matter is messy — sometimes it's dark, sometimes it's light, and the physicists are still arguing about what is there — so for the most part we ignore matter and consider simply the vacuum Einstein equations

$$G_{ab} = 0. \tag{2}$$

So we have geometry — but Lorentzian, rather than Riemannian, since $g_{ab}$ has Minkowski signature $g_{ab} \sim \mathrm{diag}(-1, 1, 1, 1)$ rather than Riemannian positive definite. Curves $s \mapsto x(s) = (x^a(s))$ are *timelike* if $g(\dot{x}, \dot{x}) = g_{ab}\dot{x}^a\dot{x}^b < 0$, *lightlike* (or *null*) if $g(\dot{x}, \dot{x}) = 0$ and *spacelike* otherwise.

## 2 Analysis

Choosing a spacelike hypersurface $M$ is a natural idea, since a globally hyperbolic spacetime (where the Cauchy problem is well-posed) is then diffeomorphic to $M \times \mathbb{R}$. A fundamental result of Y. Choquet-Bruhat [12] shows that the spacetime is determined by the Einstein equations with initial data consisting of the induced (Riemannian) metric $g_{ij}$ and the second fundamental form $K_{ij}$, since (1) becomes a hyperbolic system for the 10 functions $g_{ab}$ when a wave map gauge is imposed.

The Choquet-Bruhat existence result introduces the most interesting mathematical theme, partial differential equations. Many other applications are not hyperbolic, but elliptic or even parabolic. One simple example is the *maximal surface equation* $\mathrm{Tr}_g K = 0$, which in flat space $\mathbb{R}^{n,1}$ reduces to

$$\frac{1}{\sqrt{1-|Du|^2}}\left(\delta_{ij} + \frac{D_i u D_j u}{1-|Du|^2}\right) D^2_{ij} u = 0, \tag{3}$$

for $M = \mathrm{graph}\, u$. A priori estimates for $(1 - |Du|^2)^{-1/2}$, analogous to gradient bounds for the minimal surface equation, ensure (3) is uniformly elliptic and existence and regularity for the Dirichlet problem follows by standard arguments. This was my introduction to non-linear PDE [9]. Note that (3) has applications to wheat flow in silos [13].

The geometric initial data $(g_{ij}, K_{ij})$ are not freely specifiable; the Einstein equations imply the ubiquitous *constraint equations*

$$0 = 2G_{00} \quad = R - |K|^2 + (\mathrm{Tr}_g K)^2 \quad =: 2\Phi_0(g, K) \tag{4}$$

$$0 = G_{0i} \quad = \nabla^j K_{ij} - \nabla_i \mathrm{Tr}_g K \quad =: \Phi_i(g, K). \tag{5}$$

The scalar curvature $R$ in the Hamiltonian constraint (4) illustrates the structure of the space of solutions of the constraints: for any 3-metric $\bar{g}_{ij}$ and positive function $\phi \in C^\infty(M)$, the conformally related metric $g_{ij} = \phi^4 \bar{g}_{ij}$ has scalar curvature

$$R = R(g) = -8\phi^{-5}(\Delta_{\bar{g}}\phi - \tfrac{1}{8}R(\bar{g})\phi). \tag{6}$$

Thus $R(g)$ can be prescribed if the associated elliptic equation for $\phi$ admits a positive solution. This conformal approach (and its extension to the full system (4,5)) has been extensively studied; see [7] for a recent survey.

A parabolic construction for metrics with prescribed scalar curvature was introduced in [4]; the idea is to assume a radial foliation with metric

$$g = u^2\,\mathrm{d}r^2 + g_{AB}(\mathrm{d}x^A + b^A \mathrm{d}r)(\mathrm{d}x^B + b^B \mathrm{d}r), \tag{7}$$

where the 2-metric $g_{AB}$ and shift vector $b^A$ are given, such that the leaves have positive mean and Gauss curvature. The *second variation* identity with prescribed scalar curvature

$$R = 2D_n H + 2K_G - H^2 - \|I\!I\|^2 - 2u^{-1}\Delta u, \tag{8}$$

leads to a parabolic equation for $u$, flowing radially outward. One easy application is the existence of non-flat spacetimes which contain a flat subregion; others are given in [18, 17]. Parabolic equations also arise in the Robinson-Trautman spacetimes [11]; and in the proofs of the Penrose Conjecture [14, 10].

The ADM Hamiltonian [1]

$$\mathcal{H}_{ADM} = -\int_M \xi^a \Phi_a(g, K)\,\mathrm{d}v_g, \tag{9}$$

where the lapse-shift $\xi$ acts as a Lagrange multiplier, generates the Hamiltonian form of the evolution equations

$$\frac{d}{dt}\begin{pmatrix} g \\ K \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D\Phi_a^* \xi^a \ . \tag{10}$$

For non-compact $M$ this ignores boundary terms, which in the asymptotically flat case give the ADM (total) energy-momentum vector

$$32\pi E = \oint_{S_\infty} (\partial_j g_{ij} - \partial_i g_{jj}) n^i\,\mathrm{d}S, \tag{11}$$

$$16\pi P_i = \oint_{S_\infty} (K_{ij} - \mathrm{Tr}_g K g_{ij}) n^j\,\mathrm{d}S, \tag{12}$$

which is consistently defined for $(g, K)$ in the weighted Sobolev space $\delta + H^2_{-1/2} \times H^1_{-3/2}$ [2]. The proofs [15, 16, 19] that $(E, P_i)$ is future timelike — the celebrated Positive Mass Theorem — both rely heavily on elliptic PDE, using minimal surface theory [15, 16] and the Riemannian Dirac equation [19]. A detailed existence proof for the Witten spinor $\mathcal{D}\psi = 0$ with spectral boundary conditions is given in [6].

The ADM energy-momentum $(E, P_i)$ is only defined asymptotically and coordinate invariance means it is not possible to give a geometrically consistent expression for a local energy density for the gravitational field. The correspondence principle demands at least a *quasi-local* quantity $m_{QL}(\Omega)$ which measures the gravitational mass of a bounded region $\Omega \subset M$. Minimal desirable properties of $m_{QL}$ are

(a) positivity: $m_{QL}(\Omega) > 0$ for $\Omega$ non-flat;
(b) monotonicity: if $\Omega_1 \subset \Omega_2$ then $m_{QL}(\Omega_1) \leq m_{QL}(\Omega_2)$;
(c) compatibility with $m_{ADM} = \sqrt{E^2 - |P|^2}$: $m_{QL}(\Omega) \leq m_{ADM}(\tilde{M})$ for admissible extensions of $\Omega$, where "admissible" means satisfies the conditions of PMT and has no black holes.

In [3] I suggested

$$m_{QL}(\Omega) = \text{infimum}\{m_{ADM}(\tilde{M}) : \tilde{M} \text{ is an admissible extension of } \Omega\}, \tag{13}$$

which satisfies the basic conditions (a), (b), (c), but presents many computational difficulties and analytical questions. Extrema (if they exist) are *stationary* spacetimes i.e. admit a timelike Killing vector. The Stationary Metric Conjecture proposes that the infimum is achieved, and has PMT and the remarkable Penrose inequality [14, 10]

$$\text{Area}(\Sigma) \leq 16\pi m_{ADM}^2, \tag{14}$$

for the area of a black hole $\Sigma$ as special cases.

## 3 Numerics

Numerical simulation is important for physics, since it should be able to predict the gravitational wave signatures of astrophysical phenomena such as black hole collision, supernovae collapse, etc. Coordinate invariance also greatly complicates the task of constructing reliable algorithms, and there is still no clear picture of the optimal approach. With Andrew Norton [8] we constructed a 4th-order 4D code which models the interaction of gravitational waves with a black hole, using the space-time metric in *null quasi-spherical* form

$$\mathrm{d}s^2_{NQS} = -2u\,\mathrm{d}z\,(\mathrm{d}r + v\,\mathrm{d}z) + 2|r\Theta + \overline{\beta}\mathrm{d}r + \overline{\gamma}\mathrm{d}z|^2, \tag{15}$$

based on outgoing light rays and the corresponding null coordinate $z$. Denoting by $U$ certain first derivatives of the metric functions $(u, v, \beta, \gamma)$, the Einstein equations imply the transport equation

$$\frac{\partial}{\partial r}U = F(U, \beta) \quad \text{and} \quad \beta_z = G(U, \beta),$$

which forms the basis of the code (http://www.relativity.ise.canberra.edu.au). Analysis of the numerical results gives new insight into asymptotic structure [5].

## 4 Challenges

Finally, the current big three relativity problems:

- Extend local existence (and uniqueness??) theorems for the Einstein equations to data $(g, K) \in H^2 \times H^1$, consistent with the optimal Hamiltonian phase space.
- Prove the Stationary Metric conjecture for quasi-local mass.
- Produce a reliable numerical simulation of the inspiral and collapse of a binary black hole system.

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