Some More First-Order Recurrence Relations and their Inverses

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Cohen and Whitaker [1] investigated some first-order recurrence relations related to the Chebyshev polynomials of the first kind, $T_n(x)$, defined for non-negative integer n by

$$t_n(x) = \cos(n \cos^{-1} x), \qquad |\, x \,| \leq 1.$$

For $n \geq 2$, these satisfy the Chebyshev recurrence relation

$$T_{n+2} = 2xT_{n+1} - T_n, \quad T_0 = 1, \quad T_1 = x.$$

For example, the recurrence relation

$$X_{n+1} = 2X_n^2 - 1, \quad X_0 = X,$$

can be written as

$$X_{n+1} = T_2(X_n) = \cos(2 \cos^{-1} X_n).$$

Repeated application leads to the closed form

$$X_n = \cos(2^n \cos^{-1} x) \quad (= T_{2^n}(x)).$$

For suitably restricted x this sequence of polynomials may then be considered to be the closed-form solution of the first-order recurrence relation given above. (The trigonometric form is more stable numerically than the explicit polynomial form.)

The underlying method can be applied to other recurrence relations.

Given

$$X_{n+1} = h(X_n), \quad X_0 = x,$$

then

$$X_n = h^n(x)$$

where the superscript denotes repeated function composition. If h^n can be easily written in closed form then the recurrence relation has been “solved". Otherwise it may be possible to find functions f and g such that $h = f \circ g \circ f^{-1}$. (There is clearly an analogy here with matrix algebra in terms of equivalent matrices). If g^n is more easily expressible in closed form then a solution can be obtained by setting

$$X_{n+1} = f \circ g \circ f^{-1}(X_n)$$

whence

$$X_n = f \circ g^n \circ f^{-1}(x)$$

for suitably restricted x.

The function g can also depend on n. in this case

$$X_{n+1} = f \circ g_n \circ f^{-1}(X_n)$$

with

$$X_n = f \circ g_n \circ g_{n-1} \circ \cdots \circ g_1 \circ f^{-1}(x).$$

Of course, for a given function h, finding a useful “equivalent" function g is a matter of some art. In the following we shall consider those recurrence relations which arise from given functions f and g. Also, x is to be suitably restricted in each case for the solution to be valid, and $a \neq 0, b, c$ and d are real constants and n and p are non-negative integers.

1a. Let $f(\theta) = a\cos\theta + b$ and $g(\theta) = p\theta$. The associated recurrence relation is then

$$X_{n+1} = at_p\left(\frac{X_n - b}{a}\right) + b, \quad X_0 = x.$$

Since $g^n(\theta) = p^n\theta$, the closed-form solution is

$$X_n = a\cos\left(p^n \cos^{-1}\left(\frac{x-b}{a}\right)\right) + b.$$

Particular cases include:

$$\begin{aligned}
X_{n+1} &= 2x_n^2 - 1 \qquad (p = 2, a = 1, b = 0),\\
X_{n+1} &= x_n^2 - 2bx_n + b^2 + b - 2 \qquad (p = 2, a = 2),\\
X_{n+1} &= 4x_n^3 - 3x_n \qquad (p = 3, a = 1, b = 0), \text{ and}\\
X_{n+1} &= 8x_n^4 - 8x_n^2 - 1 \qquad (p = 4, a = 1, b = 0).
\end{aligned}$$

1b. Now let $f(\theta) = a\cos\theta + b$ and $g_n(\theta) = n\theta$. The associated recurrence relation is then

$$X_{n+1} = at_n\left(\frac{X_n - b}{a}\right) + b, \quad X_0 = x,$$

with solution

$$X_n = a\cos\left(n!\cos^{-1}\left(\frac{x-b}{a}\right)\right) + b.$$

Properties of other trigonometric functions can also be used.

2a. $W_n(x) = \tan(n\tan^{-1} x)$ defines a sequence of rational functions which satisfies

$$W_{n+1} = \frac{x + W_n}{1 - xW_n}, \quad W_0 = 0.$$

Letting $f(\theta) = a\tan\theta + b$ and $g(\theta) = p(\theta)$ gives the recurrence relation

$$X_{n+1} = aW_p\left(\frac{X_n - b}{a}\right) + b, \quad x_0 = x,$$

With solution analogous to 1a above. Particular cases include:

$$\begin{aligned}
X_{n+1} &= \frac{2X_n}{1 - X_n^2} \qquad (p = 2, a = 1, b = 0),\\
X_{n+1} &= \frac{-3X_n + X_n^3}{-1 + 3X_n^2} \qquad (p = 3, a = 1, b = 0), \text{ and}\\
X_{n=1} &= \frac{4X_n - 4X_n^3}{1 - 6X_n^2 + X_n^4} \qquad (p = 4, a = 1, b = 0).
\end{aligned}$$

2b. Generalising, the rational functions determined by

$$\hat{W}_n(x) = \tan(c + \tan^{-1} x) \qquad \left(= \frac{\tan c + W_n(x)}{1 - W_n(x) \tan c} \right)$$

satisfy the same recurrence relation as $W_n(x)$ with $\hat{W}_0 = \tan q$. with $f(\theta) = a \tan\theta + b$ and $g(\theta) = c + p\theta$, the associated recurrence relation is

$$X_{n+1} = a\hat{W}_p \left(\frac{X_n - b}{a} \right) + b, \quad X_0 = x,$$

with solution

$$X_n = a \tan \left(g^n \left(\tan^{-1} \left(\frac{x-b}{a} \right) \right) \right) + b$$

where

$$g^n(\theta) = \begin{cases} c\dfrac{p^n - 1}{p - 1} + p^n\theta & p \neq 1, \\ cn + \theta & p = 1. \end{cases}$$

Examples of recurrence relations for which closed-form solutions can be found include:

$$X_{n+1} = \frac{\tan c + X_n}{1 - X_n \tan c} \quad (p = 1, a = 1, b = 0), \text{ and}$$
$$X_{n+1} = \frac{\tan c + 2X_n - X_n^2 \tan c}{1 - 2X_n \tan c - x_n^2} \quad (p = 2, a = 1, b = 0).$$

2c. $f(\theta) = a \tan\theta + b$ and $g_n(\theta) = n\theta$ gives results analogous to 1b above. $g_n(\theta) = n + p\theta$ leads to the recurrence relation

$$X_{n+1} = a \left(\frac{\tan n + W_p\left(\frac{x_n - b}{a}\right)}{1 - W - p\left(\frac{X_n - b}{a}\right) \tan n} \right) + b, \quad X_0 = x,$$

with solution

$$X_n = a \tan \left(\hat{g}_n \left(\tan^{-1} \left(\frac{x-b}{a} \right) \right) \right) + b$$

where

$$\hat{g}_n(\theta) = g_n \circ g_{n-1} \circ \cdots \circ g_1(\theta) = \begin{cases} \dfrac{p(p^n - 1) - n(p-1)}{(p-1)^2} + p^n\theta & p \neq 1, \\ \dfrac{n}{2}(n+1) + \theta & p = 1. \end{cases}$$

Finally, trigonometric functions need not be used.

3a. $f(\theta) = a\theta^p + b$ and $g(\theta) = \theta^p$ gives the recurrence relation

$$X_{n+1} = a \left(\frac{X_n - b}{a} \right)^p + b, \quad X_0 = x,$$

and solution

$$X_n = a \left(\frac{x - b}{a} \right)^{p^n} + b.$$

Examples include:

$$X_{n+1} = X_n^2 - 2bX_n + b^2 + b \qquad (p = 2, a = 1), \text{ and}$$
$$X_{n+1} = X_n^3 + 3X_n^2 + 3X_n + 2 \qquad (p = 3, a = 1, b = -1).$$

Precisely the same recurrence relations are obtained by use of $f(\theta) = ae^\theta + b$ and $g(\theta) = p\theta$.

3b. Consider the general first-order quadratic recurrence relation

$$X_{n+1} = AX_n^2 + 2BX_n + c, \quad X_0 = x,$$

Where $A \neq 0$, B and C are real constants. This can be reduced to solving a recurrence relation of the form $X_{n+1} = X_n^2 + c$ in the following manner.

Let $f(\theta) = a\theta^2 + b$ and $g(\theta) = c + \theta^2$, then the general quadratic recurrence relation is obtained when

$$a = \frac{1}{A}, \ b = C - \frac{B^2}{A} \text{ and } c = B - B^2 + AC.$$

The solution for $n \geq 1$ is then

$$X_n = a\left(g^{n-1}\left(\frac{x-b}{a} + c\right)\right)^2 + b$$

where the $n-1$ superscript denotes function composition and $G_n = g^n(\theta)$ satisfies

$$G_{n+1} = G_n^2 + c, \quad G_0 = \theta,$$

which is of the form stated above.

3c. Let $f(\theta) = \frac{1}{\theta+d}$ and $g(\theta) = c\theta$. The associated recurrence relation is

$$X_{n+1} = \frac{X_n}{c + d(1-c)X_n}, \quad X_0 = x,$$

with closed-form solution

$$X_n = \frac{x}{c^n + d(1-c^n)x}.$$

$f(\theta) = \frac{a}{\theta+d} + b$ with $g(\theta) = c_0 + c_1 n + (c_2 + c_3 n)\theta, c_i$ integer or real as appropriate, allows a wider variety of recurrence relations. (Mathematica gives closed-form solutions for $g^n(\theta)$ when $c_0 = c_1 = 0$ or $c_2 = 0$ or $c_3 = 0$.)

For each of the recurrence relations presented in cases 1, 2 and 3 above, inverse recurrence relations can also be developed in the manner of [1] by considering

$$y_{n+1} = h^{-1}(y_n) = f \circ g_n^{-1} \circ f^{-1}(y_n), \quad y_0 = y,$$

Where g_n^{-1} is an inverse function of g, possibly chosen separately for each n. Solution methods similar to those presented above may thus be useful.

References

[1] G.L. Cohen and R.N. Whitaker, Some first-order recurrence relations and their inverses, *Austral. Math. Soc. Gazette* 22:3, August 1995, pp.124-129.

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