LIPSCHITZ FUNCTIONS WITH MAXIMAL CLARKE SUBDIFFERENTIALS ARE STAUNCH

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In a recent paper we have shown that most non-expansive Lipschitz functions (in the sense of Baire’s category) have a maximal Clarke subdifferential. In the present paper, we show that in a separable Banach space the set of non-expansive Lipschitz functions with a maximal Clarke subdifferential is not only of generic, but also staunch.

1. INTRODUCTION AND DEFINITIONS

Lipschitz functions with maximal subdifferentials provide counter-examples in non-smooth analysis and differentiability theory. In a recent paper [1], we showed that the set of Lipschitz functions with maximal subdifferentials is residual in the space of all non-expansive functions. The purpose of this note is to strengthen this by showing that, in a separable-setting the set of all non-expansive Lipschitz functions with maximal subdifferentials is not only of residual but also staunch, by which we mean the complement of the set is \( \sigma \)-porous. We now recall the appropriate notion of porosity.

Let \((Y, d)\) be a complete metric space. We denote by \(B(y, r)\) the closed ball of center \(y \in Y\) and radius \(r > 0\). A subset \(E \subset Y\) is called porous in \((Y, d)\) if there exist \(0 < \alpha \leq 1\) and \(r_0 > 0\) such that for each \(0 < r \leq r_0\) and each \(y \in Y\), there exists \(z \in Y\) for which

\[
B(z, \alpha r) \subset B(y, r) \setminus E.
\]

A subset of the space \(Y\) is called \(\sigma\)-porous in \((Y, d)\) if it is a countable union of porous subsets in \((Y, d)\). All \(\sigma\)-porous sets are of the first category. If \(Y\) is a finite dimensional Euclidean space, then \(\sigma\)-porous sets are of Lebesgue measure 0. The class of \(\sigma\)-porous sets is much smaller than the class of sets which have measure 0 and are of the first category. In fact, in each topologically complete metric space without isolated points there exists a closed nowhere dense set which is not \(\sigma\)-porous [6, Theorem 1].

Throughout, \(X\) is a separable Banach space with norm \(\| \cdot \|\), and its topological dual is denoted by \(X^*\) with dual unit ball \(B^*\). We use \(S_X\) to denote the unit sphere of \(X\).
A ⊂ X be a bounded open convex set. For a real-valued \( f : A \to R \) we say that \( f \) is \( K \)-Lipschitz on \( A \) if \( K > 0 \) and \( |f(x) - f(y)| \leq K\|x - y\| \) for all \( x, y \in A \). When \( K = 1 \), \( f \) is called nonexpansive. The Clarke derivative of \( f \) at point \( x \) in the direction \( v \) is given by
\[
f^c(x; v) := \limsup_{t \to 0} \frac{f(y + tv) - f(y)}{t},
\]
while the Clarke subdifferential \( \partial c f \) is given by:
\[
\partial c f(x) := \left\{ x^* \in X^* \mid \langle x^*, v \rangle \leq f^c(x; v) \text{ for all } v \in X \right\}.
\]
Note that \( f^c(x; v) \) is upper semicontinuous as a function of \((x, v)\). Being nonempty and weak∗ compact convex valued, the multifunction \( \partial c f : A \to 2^{X^*} \) is norm-to-weak∗ upper semicontinuous. Detailed properties about Clarke subdifferentials can be found in Clarke [3, Chapter 2], which is a sort of bible for nonsmooth analysts.

2. The Main Result

Let \( C \) be a weak∗–compact convex subset of \( X^* \). Recall that the support function of \( C \) is the function \( \sigma C : X \to R \) defined by
\[
\sigma C(v) := \sup \left\{ \langle x^*, v \rangle \mid x^* \in C \right\}.
\]
Clearly, \( \sigma C \) is sublinear, and Lipschitz with Lipschitz rate \( K := \sup \{ \|x^*\| : x^* \in C \} \).

Consider
\[
\mathcal{N}_C := \left\{ f : A \to R \mid f(x) - f(y) \leq \sigma C(x - y) \text{ for all } x, y \in A \right\}.
\]
Since each \( f \in \mathcal{N}_C \) satisfies \( f(x) - f(y) \leq K\|x - y\| \) for all \( x, y \in A \), \( \mathcal{N}_C \) is a special class of \( K \)-Lipschitz functions defined on \( A \).

For \( f, g \in \mathcal{N}_C \), set
\[
\rho(f, g) := \sup_{x \in A} |f(x) - g(x)|.
\]
One can easily verify that \((\mathcal{N}_C, \rho)\) is a complete metric space.

Our central result may now be stated.

**Theorem 1.** Assume that \( X \) is a separable Banach space and let \( A \subset X \) be a bounded open convex subset of \( X \). In the complete metric space \((\mathcal{N}_C, \rho)\), there exists a subset \( G \) such that \( \mathcal{N}_C \setminus G \) is \( \sigma \)-porous in \((\mathcal{N}_C, \rho)\), and such that each \( f \in G \) has \( \partial c f \equiv C \) on \( A \).

**Proof:** Fix \( x \in A \), \( v \in S_X \) and a natural number \( k \). Consider
\[
G(x, v, k) := \left\{ f \in \mathcal{N}_C \mid \frac{f(x + tv) - f(x)}{t} - \sigma C(v) \geq -\frac{1}{k} \text{ for some } 0 < t < \frac{1}{k} \right\}.
\]
We shall show that $\mathcal{N}_C \setminus G(x, v, k)$ is porous in $(\mathcal{N}_C, \rho)$.

According to (1), it suffices to find $0 < \alpha \leq 1$ such that for each $r \in (0, 1/k)$ and each $f \in \mathcal{N}_C$ there exists $h_2 \in \mathcal{N}_C$ for which

$$B(h_2, \alpha r) \subset B(f, r) \cap G(x, v, k).$$

Of course, here $h_2$ relies on $r$, but $\alpha$ only relies on $(x, v, k)$.

To meet this goal, we define $h : X \to R$ by

$$h(\tilde{x}) := f(x) - \frac{r}{4} + \sigma_C(\tilde{x} - x),$$

and set

$$(2) \quad h_1 := \min\{f, h\}, \quad h_2 := \max\{f - \frac{r}{2}h_1\}.$$  

Clearly, $h_2 \in \mathcal{N}_C$ and $f - r/2 \leq h_2 \leq f$, so that

$$\rho(h_2, f) \leq \frac{r}{2}.$$  

Set

$$(3) \quad \alpha := \min\{d_{X \setminus A}(x), 1\} \cdot \frac{1}{8(\sigma_C(v) + \sigma_C(-v) + 1)} \cdot \frac{1}{k}.$$  

If we let

$$(4) \quad t := \min\{d_{X \setminus A}(x), 1\} \cdot \frac{1}{4(\sigma_C(v) + \sigma_C(-v) + 1)} r,$$  

where $d_{X \setminus A}(x) := \inf\{\|x - y\| : y \in X \setminus A\}$, then $0 < t < 1/k$ and $x + tv \in A$. Note that $d_{X \setminus A}(x) > 0$ because $A$ is open and $x \in A$. Now

$$h(x + tv) = f(x) - \frac{r}{4} + t\sigma_C(v).$$

Since

$$f(x) - f(x + tv) \leq \sigma_C(-tv),$$

we have

$$f(x + tv) \geq f(x) - \sigma_C(-tv) = f(x) - t\sigma_C(v).$$

The choice of $t$ implies

$$t(\sigma_C(v) + \sigma_C(-v)) \leq \frac{r}{4},$$

so that

$$f(x) - \frac{r}{4} + t\sigma_C(v) \leq f(x) - t\sigma_C(-v).$$

It follows that $h(x + tv) \leq f(x + tv)$, and so $h_1(x + tv) = h(x + tv)$ by (2). On the other hand,

$$f(x + tv) - \frac{r}{2} \leq f(x) - \frac{r}{4} + t\sigma_C(v),$$
since $f(x + tv) - f(x) \leq \sigma_C(tv)$. Therefore, by (2),

$$h_2(x + tv) = f(x) - \frac{r}{4} + t\sigma_C(v) \quad \text{and} \quad h_2(x) = f(x) - \frac{r}{4}.$$ 

This means

$$\frac{h_2(x + tv) - h_2(x)}{t} = \sigma(v).$$

Assume that $g \in B(h_2, \alpha r)$. We shall show that $g \in G(x, v, k)$. Indeed, by (5), (4), (3),

$$\frac{g(x + tv) - g(x)}{t} - \sigma_C(v) = \frac{(g - h_2)(x + tv) - (g - h_2)(x)}{t} + \frac{h_2(x + tv) - h_2(x)}{t} - \sigma_C(v)$$

$$\geq -\frac{2\alpha r}{t} = -2\alpha rt^{-1} = -2\alpha r \left[ \frac{\min\{d_{X\setminus A}(x), 1\}}{4(\sigma_C(v) + \sigma_C(-v) + 1)} \right]^{-1}$$

$$= -\alpha \cdot \frac{8(\sigma_C(v) + \sigma_C(-v) + 1)}{\min\{d_{X\setminus A}(x), 1\}} = -\frac{1}{k}.$$ 

Therefore,

$$\{ g \in N_C : \rho(g, h_2) \leq \alpha r \} \subset G(x, v, k).$$

If $\rho(g, h_2) \leq \alpha r$, then

$$\rho(g, f) \leq \rho(g, h_2) + \rho(h_2, f) \leq \alpha r + \frac{r}{2} \leq \frac{r}{2} + \frac{r}{2} = r.$$ 

Thus

$$\{ g \in N_C : \rho(g, h_2) \leq \alpha r \} \subset \{ g \in N_C : \rho(g, f) \leq r \}.$$ 

When combined with (6), this inclusion implies that

$$N_C \setminus G(x, v, k) \quad \text{is indeed porous in} \quad (N_C, \rho).$$ 

Now let $\{x_n : n \geq 1\}$ be norm dense in $A$, $\{v_m : m \geq 1\}$ be norm dense in $S_X$. Set

$$G := \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} G(x_n, v_m, k).$$

In view of (7) and that

$$N_C \setminus G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} (N_C \setminus G(x_n, v_m, k)),$$

the set $N_C \setminus G$ must be $\sigma$-porous in $(N_C, \rho)$. If $f \in G$, then for each $x_n, v_m, k$, we have $f \in G(x_n, v_m, k)$; that is,

$$\frac{f(x_n + t_{n,m,k}v_m) - f(x_n)}{t_{n,m,k}} - \sigma_C(v_m) \geq -\frac{1}{k}. $$
for some  $0 < t_{n,m,k} < 1/k$. When $k \to \infty$, from the definition of $f^\circ$ it follows that
\[
 f^\circ(x_n; v_m) \geq \limsup_{t \downarrow 0} \frac{f(x_n + tv_m) - f(x_n)}{t} \geq \sigma_C(v_m),
\]
and consequently,
\[
 f^\circ(x_n; v_m) \geq \sigma_C(v_m) \quad \text{for all } n, m \geq 1.
\] (8)

Since $\{x_n : n \geq 1\}$ is dense in $A$ and $\{v_m : m \geq 1\}$ is dense in $S_X$, for every $x \in A$ and $v \in S_X$, we may find subsequences (without relabelling) $(x_n)$ and $(v_m)$ such that $x_n \to x$ and $v_m \to v$. By the upper semicontinuity of $f^\circ$ and continuity of $\sigma_C$, from (8) we get
\[
 f^\circ(x; v) \geq \sigma_C(v).
\] (9)

Since $f \in N_C$, for every $y \in A$, $t > 0$,
\[
 f(y + tv) - f(y) \leq \sigma_C(tv).
\]

Dividing both sides by $t$, and taking the lim sup as $y \to x$ and $t \downarrow 0$ produces
\[
 f^\circ(x; v) \leq \sigma_C(v).
\]

Together with (9), we obtain
\[
 f^\circ(x; v) = \sigma_C(v) \quad \text{for } x \in A, v \in S_X.
\]

Dually, $\partial_c f(x) = C$ for every $x \in A$, and the proof of the theorem is complete. 

Observe that
\[
 N_{B^*} := \{ f \mid f : A \to R \text{ is nonexpansive with respect to } \| \cdot \| \}.
\]

Theorem 1 gives:

**Corollary 1.** In the space of nonexpansive functions, $(N_{B^*}, \rho)$, the set
\[
 \{ f \in N_{B^*} \mid \partial_c f \equiv B^* \text{ on } A \},
\]
has a $\sigma$-porous complement in $(N_{B^*}, \rho)$.

It is well-known that every locally Lipschitz function $f$ on an open subset $A$ of a separable Banach space $X$ is Gâteaux differentiable everywhere on $A$ except for possibly a Haar-null subset. We need a result due to Giles and Sciffer [4].

**Lemma 1.** Let $f : A \to R$ be a locally Lipschitz function on an open subset $A$ of a separable Banach space $X$. Then the set
\[
 \{ x \in A \mid f^+(x; v) = f^\circ(x; v) \quad \text{for all } v \in X \},
\]
is residual in $A$. Here
\[
 f^+(x; v) := \limsup_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.
\]
Combining Corollary 1 with Lemma 1 gives the following result.

**Corollary 2.** In the space of nonexpansive functions, \((\mathcal{N}_{B^*}, \rho)\), the set 
\[ \{ f \in \mathcal{N}_{B^*} \mid f \text{ is Gâteaux differentiable at most on a first category subset of } A \} \]
has a \(\sigma\)-porous complement in \((\mathcal{N}_{B^*}, \rho)\).

**Proof:** Let \( f \in \mathcal{N}_{B^*} \) such that \( \partial_c f \equiv B^* \) on \( A \). Consider the set 
\[ S_f := \{ x \in A \mid f^+(x; v) = f^o(x; v) \text{ for all } v \in X \} . \]
By Lemma 1, \( S_f \) is a residual set in \( A \). If \( f \) is Gâteaux differentiable at \( x \), then \( f^+(x; v) = \langle \nabla f(x), v \rangle \) for every \( v \in X \), and so \( x \notin S_f \) since \( \partial_c f(x) = B^* \). Therefore, such an \( f \) is at most Gâteaux differentiable on \( A \setminus S_f \), which is a first category subset in \( A \). Since the set 
\[ \{ f \in \mathcal{N}_{B^*} \mid \partial_c f \equiv B^* \text{ on } A \} , \]
has a \(\sigma\)-porous complement in \((\mathcal{N}_{B^*}, \rho)\) by Corollary 1, the result is proved.

Finally, for various generic aspects of Lipschitz functions with maximal Clarke subdifferentials on general Banach spaces, we refer readers to [2].

**References**


