A FUNCTIONAL INEQUALITY FOR THE
POLYGAMMA FUNCTIONS

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Let
\[ \Delta_n(x) = \frac{x^{n+1}}{n!} |\psi^{(n)}(x)| \quad (x > 0; n \in \mathbb{N}), \]
where \( \psi \) denotes the logarithmic derivative of Euler’s gamma function. We prove that the functional inequality
\[ \Delta_n(x) + \Delta_n(y) < 1 + \Delta_n(z), \quad x^r + y^r = z^r, \]
holds if and only if \( 0 < r \leq 1 \). And, we show that the converse is valid if and only if \( r < 0 \) or \( r \geq n + 1 \).

1. Introduction

In 1973, Grünbaum [6] presented the following elegant inequality for the Bessel function \( J_0 \).

(1.1) \[ J_0(x) + J_0(y) \leq 1 + J_0(z), \quad x^2 + y^2 = z^2. \]

Askey [4] offered a new proof of (1.1) and showed that (1.1) can be extended to \( J_\alpha \) with \( \alpha > 0 \).

\[ J_\alpha^*(x) + J_\alpha^*(y) \leq 1 + J_\alpha^*(z), \quad x^2 + y^2 = z^2; \]

where
\[ J_\alpha^*(x) = 2^\alpha \Gamma(\alpha + 1)x^{-\alpha} J_\alpha(x). \]

It is natural to ask whether there exist other special functions which satisfy inequalities of Grünbaum-type.

The logarithmic derivative of the gamma function, \( \psi = \Gamma'/\Gamma \), is known in the literature as the digamma or psi function. Its derivatives
\[ \psi', \psi'', \psi''', \ldots \]
are called polygamma functions. We have the integral and series representations
\[
\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty e^{-xt} \frac{t^n}{1-e^{-t}} dt = (-1)^{n+1}n! \sum_{k=0}^\infty \frac{1}{(x+k)^{n+1}} \quad (x > 0; \ n \in \mathbb{N}).
\]

These functions have interesting applications in various fields. In particular, they play an important role in mathematical physics. Their main properties can be found, for instance, in [1, Chapter 6]. Inequalities for digamma and polygamma functions are discussed in [3]. We also refer to [5], where a survey on gamma function inequalities is given.

In this note, we show that the trigamma function \(\psi'\) satisfies
\[
1 + z^2 \psi'(z) < x^2 \psi'(x) + y^2 \psi'(y), \quad x^2 + y^2 = z^2.
\]
Actually, (1.3) is a special case of a more general inequality involving the function
\[
\Delta_n(x) = \frac{x^{n+1}}{n!} \left| \psi^{(n)}(x) \right| \quad (x > 0; \ n \in \mathbb{N}),
\]
which we provide in the next section.

2. Main result

To prove our theorem we need properties of \(\Delta_n\) and its derivative.

**Lemma.** Let \(n \geq 1\) be an integer. The functions \(\Delta_n\) and \(\Delta'_n\) are strictly increasing on \((0, \infty)\). Moreover,
\[
\lim_{x \to 0} \Delta_n(x) = 1 \quad \text{and} \quad \lim_{x \to 0} \Delta'_n(x) = 0.
\]

**Proof:** The monotonicity and the convexity of \(\Delta_n\) are proved in [2] and [3], respectively. Using the recurrence formula
\[
\left| \psi^{(n)}(x) \right| = \left| \psi^{(n)}(x+1) \right| + \frac{n!}{x^{n+1}}
\]
(see [1, p. 260]), we obtain
\[
\Delta_n(x) = 1 + \frac{x^{n+1}}{n!} \left| \psi^{(n)}(x+1) \right|
\]
and
\[
\Delta'_n(x) = \frac{n+1}{n!} x^n \left| \psi^{(n)}(x+1) \right| - \frac{x^{n+1}}{n!} \left| \psi^{(n+1)}(x+1) \right|.
\]

From (2.2) and (2.3) we conclude that (2.1) holds.
We are now in a position to prove (1.3) and its extension to higher derivatives.

**Theorem.** Let \( n \geq 1 \) be an integer and let \( r \neq 0 \) be a real number. The inequality
\[
\Delta_n(x) + \Delta_n(y) < 1 + \Delta_n(z)
\]
holds for all positive real numbers \( x, y, z \) with \( x^r + y^r = z^r \) if and only if \( 0 < r \leq 1 \). And,
\[
1 + \Delta_n(z) < \Delta_n(x) + \Delta_n(y)
\]
is valid for all \( x, y, z > 0 \) with \( x^r + y^r = z^r \) if and only if \( r < 0 \) or \( r \geq n + 1 \).

**Proof:** We define for \( x, y > 0 \):
\[
f_{n,r}(x,y) = 1 + \Delta_n((x^r + y^r)^{1/r}) - \Delta_n(x) - \Delta_n(y).
\]
First, we assume that \( f_{n,r}(x,y) > 0 \) for all \( x, y > 0 \). Then we obtain
\[
f_{n,r}(x,x) = 1 + \Delta_n(2^{1/r}x) - 2\Delta_n(x) > 0.
\]
The asymptotic formula
\[
\left| \psi^{(n)}(x) \right| \sim \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \cdots \quad (x \to \infty)
\]
(see [1, p. 260]), gives
\[
\lim_{x \to \infty} \frac{\Delta_n(x)}{x} = \frac{1}{n}.
\]
Thus,
\[
0 \leq \lim_{x \to \infty} \frac{f_{n,r}(x,x)}{x} = \frac{1}{n}(2^{1/r} - 2).
\]
This leads to \( 0 < r \leq 1 \).

Next, we prove that if \( 0 < r \leq 1 \), then
\[
f_{n,r}(x,y) > 0 \quad \text{for all} \quad x, y > 0.
\]
Since \( r \mapsto (x^r + y^r)^{1/r} \) is decreasing on \((0, \infty)\), we conclude from the Lemma that \( r \mapsto f_{n,r}(x,y) \) is also decreasing on \((0, \infty)\). Hence,
\[
f_{n,r}(x,y) \geq f_{n,1}(x,y) = 1 + \Delta_n(x + y) - \Delta_n(x) - \Delta_n(y) = g_n(x,y), \quad \text{say}.
\]
Applying the Lemma again we obtain
\[
\frac{\partial}{\partial x} g_n(x,y) = \Delta'_n(x + y) - \Delta'_n(x) > 0.
\]
This leads to
\[
g_n(x,y) > g_n(0,y) = 0.
\]
From (2.7) and (2.8) it follows that (2.6) holds.

Now, we consider (2.5). Let \( r > 0 \). We suppose that

\[
(2.9) \quad f_{n,r}(x,y) < 0 = f_{n,r}(0,y) \quad (x,y > 0).
\]

Partial differentiation gives

\[
(2.10) \quad 1 \frac{x}{x^n} \frac{\partial}{\partial x} f_{n,r}(x,y) = x^{r-1-n} \Delta'_{n}((x^r + y^r)^{1/r})(x^r + y^r)^{1/r-1} - \frac{\Delta'_{n}(x)}{x^n}.
\]

Formula (2.3) yields

\[
(2.11) \quad \lim_{x \to 0} \frac{\Delta'_{n}(x)}{x^n} = \frac{n+1}{n!} |\psi^{(n)}(1)|
\]

and an application of the Lemma implies

\[
(2.12) \quad \lim_{x \to 0} \Delta'_{n}((x^r + y^r)^{1/r})(x^r + y^r)^{1/r-1} = \Delta'_{n}(y)y^{1-r} > 0.
\]

From (2.9)–(2.12) we conclude that \( r - 1 - n \geq 0 \).

It remains to show that if \( r < 0 \) or \( r \geq n + 1 \), then

\[
(2.13) \quad f_{n,r}(x,y) < 0 \quad \text{for all} \quad x,y > 0.
\]

Let \( r < 0 \). We have

\[
(x^r + y^r)^{1/r} < \min(x,y),
\]

so that the Lemma implies

\[
f_{n,r}(x,y) < 1 + \Delta_{n}(\min(x,y)) - \Delta_{n}(x) - \Delta_{n}(y) < 0.
\]

Let \( r \geq n + 1 \) and

\[
s = s_{n}(x,y) = (x^{n+1} + y^{n+1})^{1/(n+1)}.
\]

We obtain

\[
(2.14) \quad f_{n,r}(x,y) \leq 1 + \Delta_{n}(s) - \Delta_{n}(x) - \Delta_{n}(y) = u_{n}(x,y), \quad \text{say}.
\]

Differentiation yields

\[
(2.15) \quad \frac{\partial}{\partial x} u_{n}(x,y) = x^{n}[v_{n}(s) - v_{n}(x)],
\]

where

\[
v_{n}(x) = \frac{\Delta'_{n}(x)}{x^n}.
\]

Using

\[
\frac{1}{x} = \int_{0}^{\infty} e^{-x t} dt \quad (x > 0),
\]
the integral representation (1.2), and the convolution theorem for Laplace transforms, we obtain

\begin{equation}
\frac{n!}{x^n} \frac{\psi'(x)}{x} = -\frac{n+2}{x} \left| \psi^{(n+1)}(x) \right| + \left| \psi^{(n+2)}(x) \right| = \int_0^\infty e^{-xt} Z_n(t) \, dt,
\end{equation}

where

\[ Z_n(t) = \frac{t^{n+2}}{1-e^{-t}} - (n+2) \int_0^t \frac{s^{n+1}}{1-e^{-s}} \, ds. \]

We have

\[ Z_n(0) = 0 \quad \text{and} \quad Z'_n(t) = -\frac{t^{n+2}e^{-t}}{(1-e^{-t})^2}. \]

This implies that \( Z_n \) is negative on \((0, \infty)\). From (2.16) we find that \( v_n \) is strictly decreasing on \((0, \infty)\). Since \( s > x \), we obtain from (2.15) that

\begin{equation}
\psi^{(n)}(x) = 0.
\end{equation}

Combining (2.14) and (2.17) we conclude that (2.13) is valid.

\[ \Box \]

References


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