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BIFURCATION OF POSITIVE ENTIRE SOLUTIONS FOR A SEMILINEAR ELLIPTIC EQUATION

Tsing-San Hsu and Huei-Li Lin

In this paper, we consider the nonhomogeneous semilinear elliptic equation

\[(*)_{\lambda} \quad -\Delta u + u = \lambda K(x)u^p + h(x) \text{ in } \mathbb{R}^N, \quad u > 0 \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),\]

where \(\lambda \geq 0\), \(1 < p < (N+2)/(N-2)\), if \(N \geq 3\), \(1 < p < \infty\), if \(N = 2\), \(h(x) \in H^{-1}(\mathbb{R}^N)\), \(0 \neq h(x) \geq 0\) in \(\mathbb{R}^N\), \(K(x)\) is a positive, bounded and continuous function on \(\mathbb{R}^N\). We prove that if \(K(x) \geq K_\infty > 0\) in \(\mathbb{R}^N\), and \(\lim_{|x| \to \infty} K(x) = K_\infty\), then there exists a positive constant \(\lambda^*\) such that \((*)_{\lambda}\) has at least two solutions if \(\lambda \in (0, \lambda^*)\) and no solution if \(\lambda > \lambda^*\). Furthermore, \((*)_{\lambda}\) has a unique solution for \(\lambda = \lambda^*\) provided that \(h(x)\) satisfies some suitable conditions. We also obtain some further properties and bifurcation results of the solutions of \((*)_{\lambda}\) at \(\lambda = \lambda^*\).

1. Introduction

In this paper, we consider the semilinear elliptic equation

\[(1.1)_\lambda \quad \begin{cases} -\Delta u + u = \lambda K(x)u^p + h(x) \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \end{cases}\]

where \(\lambda \geq 0\), \(1 < p < (N+2)/(N-2)\), if \(N \geq 3\), \(1 < p < \infty\), if \(N = 2\), \(h(x) \in H^{-1}(\mathbb{R}^N)\), \(0 \neq h(x) \geq 0\) in \(\mathbb{R}^N\), \(K(x)\) is a positive, bounded and continuous function on \(\mathbb{R}^N\). Moreover, \(h(x)\) and \(K(x)\) satisfy the following conditions:

- \((h1)\) \(h(x) \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)\) for some \(q > N/2\) if \(N \geq 3\), \(q = 2\) if \(N = 2\).
- \((k1)\) \(K(x) \geq K_\infty > 0\) in \(\mathbb{R}^N\), and \(\lim_{|x| \to \infty} K(x) = K_\infty\).

The homogeneous case, that is, \(h(x) \equiv 0\), the equation \((1.1)_\lambda\) has been studied by many authors (see \([5, 8, 13, 14, 15]\)). For the nonhomogeneous case \((h(x) \neq 0)\), Zhu \([16]\), Zhu and Zhou \([18]\) and Cao and Zhou \([6]\), established the existence of multiple positive solutions of equations with structure unlike that here.
The main aim of this paper is concerned with the existence and nonexistence of multiple positive solutions of \((1.1)_\lambda\) for the full \(\lambda \in [0, \infty)\). We also obtain some properties of solutions and some bifurcation results of solutions at \(\lambda = 0\) and \(\lambda = \lambda^*\), where \(\lambda^*\) is given in Theorem 1.1 below.

Throughout this paper, we always assume that \(h(x) \geq 0, h(x) \neq 0\) in \(\mathbb{R}^N\), \(K(x)\) is a positive, bounded and continuous function on \(\mathbb{R}^N\) and \(u_0\) is the unique solution of \((1.1)_0\), unless otherwise specified and we set

\[
||u|| = \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx \right)^{1/2},
\]

\[
||u||_q = \left( \int_{\mathbb{R}^N} |u|^q \, dx \right)^{1/q}, \quad 2 \leq q < \infty,
\]

\[
||u||_\infty = \sup_{x \in \mathbb{R}^N} |u(x)|,
\]

\[
M = \inf \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx : \int_{\mathbb{R}^N} |u|^{p+1} \, dx = 1 \right\}.
\]

Now, we state our main results in the following.

**Theorem 1.1.** If \(h(x) \geq 0\) and \(h(x) \neq 0\) in \(\mathbb{R}^N\), \(K(x)\) is a positive, bounded and continuous function on \(\mathbb{R}^N\) and \(K(x)\) satisfies \((k1)\). Then there is \(\lambda^*, 0 < \lambda^* < \infty\), such that:

(i) \((1.1)_\lambda\) has at least two solutions \(u_\lambda, U_\lambda\) and \(u_\lambda < U_\lambda\) if \(\lambda \in (0, \lambda^*)\);

(ii) \((1.1)_{\lambda^*}\) has a unique solution \(u_{\lambda^*}\) provided that \(h(x)\) satisfies \((h1)\);

(iii) \((1.1)_\lambda\) has no positive solutions if \(\lambda > \lambda^*\).

Furthermore,

\[
\lambda_1 \equiv \frac{(p + 1)(p - 1)^{p-1}M^{(p+1)/2}}{(2p)^p ||K||_\infty ||h||_{H^{-1}}^{p-1}} \leq \frac{\lambda^* \leq \inf_{w \in H^1(\mathbb{R}^N)\setminus\{0\}} \left( \frac{||w||^2}{p \int_{\mathbb{R}^N} K u_0^{p-1} w^2 \, dx} \right)}{p ||h||_{H^{-1}}^2 \int_{\mathbb{R}^N} K u_0^{p+1} \, dx} \equiv \lambda_2
\]

where \(u_\lambda\) is the minimal solution of \((1.1)_\lambda\), \(U_\lambda\) is the second solution of \((1.1)_\lambda\) constructed in Section 4 and \(u_0\) is the unique positive solution of \((1.1)_0\).

**Theorem 1.2.** If \((h1), (k1)\) hold, \(h(x) \geq 0\), \(h(x) \neq 0\) in \(\mathbb{R}^N\) and \(K(x)\) is a positive, bounded and continuous function on \(\mathbb{R}^N\). Then

(i) \(u_\lambda\) is strictly increasing with respect to \(\lambda\), \(u_\lambda\) is uniformly bounded in \(L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)\) for all \(\lambda \in [0, \lambda^*]\) and

\[
u_\lambda \to u_0 \text{ in } L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \text{ as } \lambda \to 0,
\]
where $u_0$ is the unique positive solution of (1.1)$_0$.

(ii) $U_\lambda$ is strictly decreasing with respect to $\lambda$ and $U_\lambda$ is unbounded in $L^\infty(\mathbb{R}^N)$ \(\cap H^1(\mathbb{R}^N)\), that is
\[
\lim_{\lambda \to 0} \|U_\lambda\| = \lim_{\lambda \to 0} \|U'_\lambda\|_\infty = \infty.
\]

(iii) Moreover, we assume that $K(x)$ and $h(x)$ are in $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, then all solutions of (1.1)$_\lambda$ are in $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$, and $(\lambda^*, u_{\lambda^*})$ is a bifurcation point for (1.1)$_\lambda$ and
\[
u_\lambda \to u_0 \text{ in } C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \text{ as } \lambda \to 0,
\]
where $u_0$ is the unique positive solution of (1.1)$_0$.

We shall organise this paper as follows. In Section 2, we give some notations and preliminary results. In Section 3, we assert that there exists $\lambda^* > 0$ such that (1.1)$_\lambda$ has a minimal solution for $\lambda \in [0, \lambda^*)$. In Section 4, we establish the existence of a second solution $U_\lambda$ for $\lambda \in (0, \lambda^*)$ and some asymptotic behaviour of the solution of (1.1)$_\lambda$. In Section 5, we shall give some further properties, and bifurcation of solutions of (1.1)$_\lambda$.

2. Preliminaries

In this section, we shall give some notations and some known results. In order to get the existence of positive solutions of (1.1)$_\lambda$, we consider the energy functional $I_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by
\[
I_\lambda(u) = \int_{\mathbb{R}^N} \left[ \frac{1}{2} (|\nabla u|^2 + |u|^2) - \frac{\lambda}{p+1} K(x)(u^+)^{p+1} - h(x)u \right] dx,
\]
where $u^\pm(x) = \max\{\pm u(x), 0\}$. Then the critical points of $I_\lambda$ are the positive solutions of (1.1)$_\lambda$. Consider the equation
\[
(2.1)_\lambda \left\{ \begin{array}{l}
-\Delta u + u = \lambda K_\infty u^p \text{ in } \mathbb{R}^N, \\
u > 0 \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),
\end{array} \right.
\]

and its associated energy functional $I_\infty^\lambda$ defined by
\[
I_\infty^\lambda(u) = \int_{\mathbb{R}^N} \left[ \frac{1}{2} (|\nabla u|^2 + |u|^2) - \frac{\lambda}{p+1} K_\infty (u^+)^{p+1} \right] dx, \quad u \in H^1(\mathbb{R}^N).
\]

It is well known that equation (2.1)$_\lambda$ has a unique ground state solution $\omega_\lambda$ and $I_\infty^\lambda(\omega_\lambda) = \sup_{t > 0} I_\infty^\lambda(t\omega_\lambda)$ (see Bahri and Lions [3] and the references there).

Now, we given the following known propositions for later use.
Proposition 2.1. Let \( K(x) \) satisfy (k1) and \( \{u_k\} \) be a \((PS)_c\)-sequence of \( I_\lambda \) in \( H^1(\mathbb{R}^N) \):

\[
I_\lambda(u_k) = c + o(1) \text{ as } k \to \infty, \\
I'_\lambda(u_k) = o(1) \text{ strongly in } H^{-1}(\mathbb{R}^N).
\]

Then there exist an integer \( l \geq 0 \), sequence \( \{x^i_k\} \subseteq \mathbb{R}^N \), functions \( u \in H^1(\mathbb{R}^N) \), \( u_i \in H^1(\mathbb{R}^N) \), \( 1 \leq i \leq l \), such that for some subsequence \( \{u_k\} \), we have

\[
\begin{cases}
u_k - \left( \sum_{i=1}^l \frac{1}{\lambda} (\cdot - x^i_k) \right) \to 0, \text{ as } k \to \infty, \\
u_k \rightharpoonup \bar{u} \text{ weakly in } H^1(\mathbb{R}^N); \\
c = I_\lambda(\bar{u}) + \sum_{i=1}^l I^\infty_\lambda(\bar{u}_i); \\
-\Delta \bar{u} + \bar{u} = \lambda K(x) \bar{u} + h(x) \text{ in } H^{-1}(\mathbb{R}^N); \\
-\Delta \bar{u}_i + \bar{u}_i = \lambda K_\infty \bar{u}_i \text{ in } H^{-1}(\mathbb{R}^N), \ 1 \leq i \leq l; \\
|x^i_k| \to \infty, \ |x^i_k - x^j_k| \to \infty, \ 1 \leq i \neq j \leq l.
\end{cases}
\]

where we agree that in the case \( l = 0 \) the above holds without \( \bar{u}_i, x^i_k \).

**Proof:** The proof can be obtained by using the arguments in Bahri and Lions [3] (also see [13, 14]). We omit it.

### 3. Existence of Minimal Solution and Decay

In this section, by the barrier method, we prove that the existence of minimal positive solution \( u_\lambda \) for all \( \lambda \) in some finite interval \( [0, \lambda^*] \) (that is, for any positive solution \( u \) of \((1.1)_\lambda \), then \( u \geq u_\lambda \)). Furthermore, we establish a decay estimate for solutions of \((1.1)_\lambda \).

**Lemma 3.1.** Let \( K(x) \) satisfy (k1). Then \((1.1)_\lambda \) has a solution \( u_\lambda \) if \( 0 \leq \lambda < \lambda_1 \) where \( \lambda_1 \) is given by (1.2).

**Proof:** For \( \lambda = 0 \), the existence question is equivalent to the existence of \( u_0 \in H^1(\mathbb{R}^N) \) such that

\[
\int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla \phi + u_0 \phi = \int_{\mathbb{R}^N} h\phi
\]

for all \( \phi \in H^1(\mathbb{R}^N) \). Now, we have that

\[
\left| \int_{\mathbb{R}^N} h\phi \right| \leq \|h\|_{H^{-1}} \|\phi\|.
\]

According to the Lax–Milgram theorem, there exists a unique \( u_0 \in H^1(\mathbb{R}^N) \) satisfies (3.1). Since \( 0 \neq h \geq 0 \) in \( \mathbb{R}^N \), by strong maximum principle (see Gilbarg and Trudinger [10]), we conclude that \( u_0 > 0 \) in \( \mathbb{R}^N \).
We consider next the case $\lambda > 0$. We show first that for sufficiently small $\lambda$, say $\lambda = \lambda_0$, there exists $t_0 = t(\lambda_0) > 0$ such that $I_{\lambda_0}(u) > 0$ for $\|u\| = t_0$. From the definitions of $I_\lambda$, we have

$$I_\lambda(u) \geq \frac{1}{2} \|u\|^2 - \frac{\lambda}{p+1} \|K\|_\infty M^{-(p+1)/2} \|u\|^{p+1} - \|h\|_{H^{-1}} \|u\|$$

Set

$$f(t) = \frac{1}{2} t - \lambda c_1 t^p - c_2,$$

where $c_1 = \|K\|_\infty/(p+1)M^{-(p+1)/2}$ and $c_2 = \|h\|_{H^{-1}}$.

It then follows that $f(t)$ achieves a maximum at $t_\lambda = (2p\lambda c_1)^{-1}$. Set

$$B_\lambda = \{u \in H^1(\mathbb{R}^N) : \|u\| < t_\lambda\}.$$ 

Then for all $u \in \partial B_\lambda = \{u \in H^1(\mathbb{R}^N) : \|u\| = t_\lambda\}$,

$$I_\lambda(u) \geq t_\lambda f(t_\lambda) \geq t_\lambda [t_\lambda (p-1)/2p - c_2] > 0$$

provided that $\lambda < \lambda_1$ which $\lambda_1$ is given by (1.2). Fix such a value of $\lambda$, say $\lambda_0$, and set $t_0 = t(\lambda_0)$. Let $0 \neq \phi \geq 0$, $\phi \in C_0^\infty(\mathbb{R}^N)$ such that \(\int_{\mathbb{R}^N} h\phi \, dx > 0\). Then

$$I_{\lambda_0}(t\phi) = \frac{t^2}{2} \|\phi\|^2 - \frac{\lambda_0}{p+1} t^{p+1} \int_{\mathbb{R}^N} K\phi^{p+1} - t \int_{\mathbb{R}^N} h\phi < 0$$

for sufficiently small $t > 0$, and it is easy to see that $I_{\lambda_0}$ is bounded below on $B_{t_0}$. Set $\alpha = \inf \{I_{\lambda_0}(u) \mid u \in B_{t_0}\}$. Then $\alpha < 0$, and since $I_{\lambda_0}(u) > 0$ on $\partial B_{t_0}$, the continuity of $I_{\lambda_0}$ on $H^1(\mathbb{R}^N)$ implies that there exists $0 < t_1 < t_0$ such that $I_{\lambda_0}(u) > \alpha$ for all $u \in H^1(\mathbb{R}^N)$ and $t_1 \leq \|u\| \leq t_0$. By the Ekeland’s variational principle [9], there exists a sequence $\{u_k\}_{k=1}^\infty \subset B_{t_1}$ such that $I_{\lambda_0}(u_k) = \alpha + o(1)$ and $I'_{\lambda_0}(u_k) = o(1)$ strongly in $H^{-1}(\mathbb{R}^N)$, as $k \to \infty$. By Proposition 2.1, we have that there exist a subsequence $\{u_k\}$, an integer $l \geq 0$, $\omega_i > 0$, $1 \leq i \leq l$ (if $l \geq 1$), $\overline{u} > 0$ in $\mathbb{R}^N$ and $\overline{u}$ in $\overline{B}_{t_1}$ such that

$$\begin{cases}
u_k \rightharpoonup \overline{u} \text{ weakly in } H^1(\mathbb{R}^N), \\
-\Delta \overline{u} + \overline{u} = \lambda_0 K(x)\overline{u}^p + h(x) \text{ in } H^{-1}(\mathbb{R}^N), \\
-\Delta \omega_i + \omega_i = \lambda_0 K_\infty \omega_i^p \text{ in } H^{-1}(\mathbb{R}^N), \quad 1 \leq i \leq l.
\end{cases}$$

Moreover,

$$I_{\lambda_0}(u_k) = I_{\lambda_0}(\overline{u}) + \sum_{i=1}^l I_{\lambda_0}^\infty(\omega_i) + o(1) \text{ as } k \to \infty.$$ 

Note that $I_{\lambda_0}^\infty(\omega_i) = I_{\lambda_0}^\infty(\omega_\lambda) > 0$ for $i = 1, 2, \ldots , l$. Since $\overline{u} \in B_{t_0}$, we have $I_{\lambda_0}(\overline{u}) \geq \alpha$. We conclude that $l = 0$, $I_{\lambda_0}(\overline{u}) = \alpha$ and $I'_{\lambda_0}(\overline{u}) = 0$, that is, $\overline{u}$ is a weak positive solution of (1.1)$_{\lambda_0}$.
Now, by the standard barrier method, we get the following Lemma.

**Lemma 3.2.** Let $K(x)$ satisfy (k1). Then there exists $\lambda^* > 0$ such that for each $\lambda \in [0, \lambda^*)$, problem $(1.1)_\lambda$ has a minimal positive solution $u_\lambda$ and $u_\lambda$ is strictly increasing in $\lambda$.

**Proof:** Denoting
\[
\lambda^* = \sup\{\lambda \geq 0 : (1.1)_\lambda \text{ has a positive solution} \}
\]
By Lemma 3.1, we have $\lambda^* > 0$. Now, consider $\lambda \in [0, \lambda^*)$. By the definition of $\lambda^*$, we know that there exists $\lambda' > \lambda$ such that $\lambda' < \lambda^*$ and $(1.1)_{\lambda'}$ has a positive solution $u_{\lambda'} > 0$, that is,
\[
-\Delta u_{\lambda'} + u_{\lambda'} = \lambda'K(x)u_{\lambda'}^p + h(x) > \lambda K(x)u_{\lambda'}^p + h(x).
\]
Then $u_{\lambda'}$ is a supersolution of $(1.1)_{\lambda}$. From $h(x) \geq 0$ and $h(x) \neq 0$, it is easily proved that 0 is a subsolution of $(1.1)_{\lambda}$. By the standard barrier method, there exists a solution $u_{\lambda}$ of $(1.1)_{\lambda}$ such that $0 \leq u_{\lambda} \leq u_{\lambda'}$. Since 0 is not a solution of $(1.1)_{\lambda}$ and $\lambda' > \lambda$, the maximum principle implies that $0 < u_{\lambda} < u_{\lambda'}$. Again using a result of Amann [1], we can choose a minimum positive solution $u_{\lambda}$ of $(1.1)_{\lambda}$. This completes the proof of Lemma 3.2.

Now, we consider a solution $u$ of $(1.1)_{\lambda}$. Let $\sigma_{\lambda}(u)$ be defined by
\[
(3.2) \quad \sigma_{\lambda}(u) = \inf\left\{ \int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2) \, dx : w \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} pKu^{p-1}w^2 \, dx = 1 \right\}
\]
By the standard direct minimisation procedure, we can show that $\sigma_{\lambda}(u)$ is attained by a function $\varphi_{\lambda} > 0$, $\varphi_{\lambda} \in H^1(\mathbb{R}^N)$, satisfying
\[
(3.3) \quad -\Delta \varphi_{\lambda} + \varphi_{\lambda} = \sigma_{\lambda}(u)pKu^{p-1}\varphi_{\lambda} \text{ in } \mathbb{R}^N.
\]

**Lemma 3.3.** Let $K(x)$ satisfy (k1). For $\lambda \in [0, \lambda^*)$, let $u_{\lambda}$ be the minimal solution of $(1.1)_{\lambda}$ and $\sigma_{\lambda}(u_{\lambda})$ be the corresponding number given by (3.2). Then
(i) $\sigma_{\lambda}(u_{\lambda}) > \lambda$ and is strictly decreasing in $\lambda$, $\lambda \in [0, \lambda^*)$;
(ii) $\lambda^* < \infty$, and $(1.1)_{\lambda^*}$ has a minimal solution $u_{\lambda^*}$.

**Proof:** Consider $u_{\lambda'}$, $u_{\lambda}$, where $\lambda^* > \lambda' > \lambda \geq 0$. Let $\varphi_{\lambda}$ be a minimiser of $\sigma_{\lambda}(u_{\lambda})$, then by Lemma 3.2, we obtain that
\[
\int_{\mathbb{R}^N} pKu_{\lambda'}^{p-1}\varphi_{\lambda}^2 \, dx > \int_{\mathbb{R}^N} pKu_{\lambda}^{p-1}\varphi_{\lambda}^2 \, dx = 1,
\]
and there is \( t, 0 < t < 1 \) such that
\[
\int_{\mathbb{R}^N} pKu_{\lambda'}^{p-1}(t\varphi_{\lambda})^2 = 1.
\]
Therefore,
\[
\sigma_{\lambda'}(u_{\lambda'}) \leq t^2\|\varphi_{\lambda}\|^2 < \|\varphi_{\lambda}\|^2 = \sigma_{\lambda}(u_{\lambda}),
\]
showing the monotonicity of \( \sigma_{\lambda}(u_{\lambda}), \lambda \in [0, \lambda^*) \).

Consider now \( \lambda \in (0, \lambda^*) \). Let \( \lambda < \lambda' < \lambda^* \). From (3.3) and the monotonicity of \( u_{\lambda} \), we get
\[
\sigma_{\lambda}(u_{\lambda}) p \int_{\mathbb{R}^N} (u_{\lambda'} - u_{\lambda}) K u_{\lambda}^{p-1} \varphi_{\lambda} \, dx
\]
\[
= \int_{\mathbb{R}^N} \nabla(u_{\lambda'} - u_{\lambda}) \cdot \nabla \varphi_{\lambda} \, dx + \int_{\mathbb{R}^N} (u_{\lambda'} - u_{\lambda}) \varphi_{\lambda} \, dx
\]
\[
= (\lambda' - \lambda) \int_{\mathbb{R}^N} K u_{\lambda'}^{p} \varphi_{\lambda} \, dx + \lambda \int_{\mathbb{R}^N} K(u_{\lambda'}^{p} - u_{\lambda}^{p}) \varphi_{\lambda} \, dx
\]
\[
\geq \lambda p \left( \int_{\mathbb{R}^N} K \varphi_{\lambda} \int_{u_{\lambda}}^{u_{\lambda'}} t^{p-1} \, dt \, dx \right)
\]
which implies that \( \sigma_{\lambda}(u_{\lambda}) > \lambda, \lambda \in (0, \lambda^*) \). This completes the proof of (i).

We show next that \( \lambda^* < \infty \). Let \( \lambda_0 \in (0, \lambda^*) \) be fixed. For any \( \lambda \geq \lambda_0 \), (3.4) and (3.5) imply
\[
\sigma_{\lambda_0}(u_{\lambda_0}) \geq \sigma_{\lambda}(u_{\lambda}) > \lambda
\]
for all \( \lambda \in [\lambda_0, \lambda^*) \). Thus, \( \lambda^* < \infty \).

By (3.2) and \( \sigma_{\lambda}(u_{\lambda}) > \lambda \), we have
\[
\int_{\mathbb{R}^N} (|\nabla u_{\lambda}|^2 + |u_{\lambda}|^2) \, dx - \lambda p \int_{\mathbb{R}^N} K u_{\lambda}^{p+1} \, dx > 0.
\]
and
\[
\int_{\mathbb{R}^N} (|\nabla u_{\lambda}|^2 + |u_{\lambda}|^2) \, dx - \int_{\mathbb{R}^N} \lambda K u_{\lambda}^{p+1} \, dx - \int_{\mathbb{R}^N} h u_{\lambda} = 0.
\]
Thus
\[
\int_{\mathbb{R}^N} (|\nabla u_{\lambda}|^2 + |u_{\lambda}|^2) \, dx = \int_{\mathbb{R}^N} \lambda K u_{\lambda}^{p+1} \, dx + \int_{\mathbb{R}^N} h u_{\lambda} \, dx
\]
\[
\leq \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u_{\lambda}|^2 + |u_{\lambda}|^2) \, dx + \|h\|_{L^2(\mathbb{R}^N)} \|u_{\lambda}\|
\]
\[
\leq \left( \frac{1}{p} + \frac{\delta}{2} \right) \|u_{\lambda}\|^2 + \frac{1}{2\delta} \|h\|^2_{L^2(\mathbb{R}^N)},
\]
for any \( \delta > 0 \). Since \( p > 1 \), we can obtain that \( \|u_{\lambda}\| \leq c < +\infty \) for all \( \lambda \in (0, \lambda^*) \) by taking \( \delta \) small enough. By Lemma 3.2, the solution \( u_{\lambda} \) is strictly increasing with respect to \( \lambda \); we may suppose that
\[
u_{\lambda} \rightharpoonup u_{\lambda^*} \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } \lambda \to \lambda^*,
\]
and hence $u_{\lambda^*}$ is a minimal solution of $(1.1)_{\lambda^*}$. This completes the proof of Lemma 3.3.

**Lemma 3.4.** If $K(x)$ satisfies (k1), then $\lambda_1 \leq \lambda^* \leq \lambda_2 \leq \lambda_3$, where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are given by (1.2).

**Proof:** By Lemma 3.1 and the definition of $\lambda^*$, we conclude that $\lambda^* \geq \lambda_1$.

As in Lemma 3.3, we have $\sigma_\lambda(u_{\lambda}) > \lambda$ for all $\lambda \in (0, \lambda^*)$, so for any $w \in H^1(\mathbb{R}^N) \setminus \{0\}$, we have

\begin{equation}
\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2) \, dx > \lambda p \int_{\mathbb{R}^N} K u_{\lambda}^{p-1} w^2 \, dx.
\end{equation}

Let $u_0$ be the unique solution of $(1.1)_0$, then by (3.6) and $u_{\lambda} > u_0$ for all $\lambda \in (0, \lambda^*)$, we obtain that

\begin{equation}
\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2) \, dx > \lambda p \int_{\mathbb{R}^N} K u_0^{p-1} w^2 \, dx,
\end{equation}

that is,

\begin{equation}
\lambda \leq \inf_{w \in H^1(\mathbb{R}^N) \setminus \{0\}} \left( \frac{|w|^2}{p \int_{\mathbb{R}^N} K u_0^{p-1} w^2 \, dx} \right) = \lambda_2.
\end{equation}

This implies that $\lambda^* \leq \lambda_2$.

For all $\lambda \in [0, \lambda^*]$, let $u_{\lambda}$ is a minimal solution of $(1.1)_{\lambda}$ and take $w = u_{\lambda}$ in (3.6), then we have that

\begin{equation}
\|u_{\lambda}\|^2 = \lambda \int_{\mathbb{R}^N} K u_{\lambda}^{p+1} \, dx + \int_{\mathbb{R}^N} h u_{\lambda} \, dx
\leq \frac{1}{p} \|u_{\lambda}\|^2 + \|h\|_{H^{-1}} \|u_{\lambda}\|.
\end{equation}

This implies that

\begin{equation}
\|u_{\lambda}\| \leq \frac{p}{p-1} \|h\|_{H^{-1}}.
\end{equation}

Take $w = u_{\lambda}$ in (3.7), and by (3.8) and the monotonicity of $u_{\lambda}$, we get that

\begin{equation}
\lambda_2 \leq \frac{\|u_{\lambda}\|^2}{p \int_{\mathbb{R}^N} K u_0^{p-1} u_{\lambda}^2 \, dx}
\leq \frac{\|h\|^2_{H^{-1}}}{(p-1)^2 \int_{\mathbb{R}^N} K u_0^{p+1} \, dx} = \lambda_3.
\end{equation}

Finally, we establish the decay estimate for solutions of $(1.1)_{\lambda}$ and this result will be used in Section 4 and Section 5. Now, we quote two Regularity Lemmas (see Hsu [11] for the proof).

**Lemma 3.5.** Let $f : \mathbb{X} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for almost every $x \in \mathbb{X}$, there holds

\begin{equation}
|f(x, u)| \leq c(|u| + |u|^p)
\end{equation}

uniformly in $x \in \mathbb{X}$,
where $\mathcal{X}$ is a $C^{1,1}$ domain in $\mathbb{R}^N$, $1 < p < (N + 2)/(N - 2)$ if $N \geq 3$, $1 < p < \infty$ if $N = 2$. Also, let $u \in H^1_0(\mathcal{X})$ be a weak solution of equation $-\Delta u = f(x, u) + h(x)$ in $\mathcal{X}$, where $h \in L^{N/2}(\mathcal{X}) \cap L^2(\mathcal{X})$. Then $u \in L^q(\mathcal{X})$ for $q \in [2, \infty)$.

**Lemma 3.6.** Let $\mathcal{X}$ be a $C^{1,1}$ domain in $\mathbb{R}^N$, $g \in L^2(\mathcal{X}) \cap L^q(\mathcal{X})$ for some $q \in [2, \infty)$ and $u \in H^1_0(\mathcal{X})$ be a weak solution of the equation $-\Delta u + u = g$ in $\mathcal{X}$. Then $u \in W^{2,q}(\mathcal{X})$ satisfies

$$\|u\|_{W^{2,q}(\mathcal{X})} \leq c\left(\|u\|_{L^q(\mathcal{X})} + \|g\|_{L^q(\mathcal{X})}\right),$$

where $c = c(N, q, \partial \mathcal{X})$.

**Lemma 3.7.** Let $h(x)$ satisfy (h1) and $u$ be a weak solution of \((1.1)_\lambda\), then

(i) $u(x) \to 0$ as $|x| \to \infty$.

(ii) there exists positive constant $c_1$ such that

\begin{equation}
(3.10) \quad u(x) \geq c_1 \exp(-|x|)|x|^{-(N-1)/2} \text{ as } |x| \to \infty.
\end{equation}

**Proof:** (i) Let $u$ satisfy

$$-\Delta u + u = \lambda K(x)u^p + h(x) \quad \text{in } H^{-1}(\mathbb{R}^N).$$

Since $K$ is bounded and $h \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for some $q > N/2$. Hence

$$h \in L^2(\mathbb{R}^N) \cap L^{N/2}(\mathbb{R}^N)$$

and by Lemma 3.5, we have $u \in L^q(\mathbb{R}^N)$ for $q \in [2, \infty)$. Hence

$$\lambda K(x)u^p + h(x) \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$$

for some $q > N/2$. Then by Lemma 3.6, we have $u \in W^{2,q}(\mathbb{R}^N)$ for some $q > N/2$. By the Sobolev embedding theorem, $u \in C^0_b(\mathbb{R}^N)$ and there exists $c > 0$, such that for any $r > 1$,

$$\|u\|_{L^\infty(\overline{B_r})} \leq c\|u\|_{W^{2,q}(\overline{B_r})},$$

where

$$\overline{B_r} = \{x \in \mathbb{R}^N : |x| > r\}.$$

Hence $\lim_{|x| \to \infty} u(x) = 0$.

(ii) It is very easy to show that $(1 + 1/\sqrt{|x|})e^{-|x|/|x|^{(N-1)/2}}$ is a subsolution of \((1.1)_\lambda\) for all $|x|$ large. Therefore (3.10) is proved by means of the maximum principle.
Lemma 3.8. Let $u_\lambda$ be the minimal solution of $(1.1)_\lambda$ for $\lambda \in [0, \lambda^*]$ and $\sigma_\lambda(u_\lambda) > \lambda$. Then for any $g(x) \in H^{-1}(\mathbb{R}^N)$, problem

$$(3.11)_\lambda \quad -\Delta w + w = \lambda p K u_\lambda^{p-1} w + g(x), w \in H^1(\mathbb{R}^N)$$

has a solution.

Proof: Consider the functional

$$\Phi(w) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) \, dx - \frac{1}{2} \lambda p \int_{\mathbb{R}^N} K u_\lambda^{p-1} w \, dx - \int_{\mathbb{R}^N} g(x) w \, dx,$$

where $w \in H^1(\mathbb{R}^N)$. From Hölder inequality and Young’s inequality, we have, for any $\varepsilon > 0$, that

$$(3.12) \quad \Phi(w) \geqslant \frac{1}{2}(1 - \lambda \sigma_\lambda(u_\lambda)^{-1})\|w\|^2 - \frac{1}{2}\varepsilon\|w\|^2 - C \varepsilon \frac{1}{2} \|g\|_{H^{-1}(\mathbb{R}^N)}^2$$

if we choose $\varepsilon$ small.

Now, let $\{w_n\} \subset H^1(\mathbb{R}^N)$ be the minimising sequence of variational problem

$$d = \inf \{\Phi(w) \mid w \in H^1(\mathbb{R}^N)\}.$$

From (3.12) and $\sigma_\lambda(u_\lambda) > \lambda$, we can also deduce that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$, if we choose $\varepsilon$ small. So we may suppose that

$$w_n \rightharpoonup w \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty,$$

$$w_n \to w \text{ almost everywhere in } \mathbb{R}^N \text{ as } n \to \infty.$$

By Fatou’s Lemma,

$$\|w\|^2 \leqslant \liminf \|w_n\|^2.$$

By Lemma 3.7, we have that $u_\lambda(x) \to 0$ as $|x| \to \infty$ and the weak convergence imply

$$\int_{\mathbb{R}^N} g w_n \, dx \to \int_{\mathbb{R}^N} g w \, dx, \quad \int_{\mathbb{R}^N} K u_\lambda^{p-1} w_n \, dx \to \int_{\mathbb{R}^N} K u_\lambda^{p-1} w \, dx \text{ as } n \to \infty.$$

Therefore

$$\Phi(w) \leqslant \lim_{n \to \infty} \Phi(w_n) = d,$$

and hence $\Phi(w) = d$ which gives that $w$ is a solution of $(3.11)_\lambda$.

Remark 3.9. From Lemma 3.8, we know that $(3.11)_\lambda$ has a solution $w \in H^1(\mathbb{R}^N)$. Now, we also assume that $K(x), h(x),$ and $g(x)$ are in $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, then by the elliptic regular theory ([10]), we can deduce that $w \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$. 


Lemma 3.10. Suppose $u_{\lambda^*}$ is a solution of $(1.1)_{\lambda^*}$, then $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$ and the solution $u_{\lambda^*}$ is unique.

**Proof:** Define $F : \mathbb{R} \times H^1(\mathbb{R}^N) \longrightarrow H^{-1}(\mathbb{R}^N)$ by

$$F(\lambda, u) = \Delta u - u + \lambda K(u^+)^p + h(x).$$

Since $\sigma(\lambda) \geq \lambda$ for $\lambda \in (0, \lambda^*)$, so $\sigma_{\lambda^*}(u_{\lambda^*}) \geq \lambda^*$. If $\sigma_{\lambda^*}(u_{\lambda^*}) > \lambda^*$, the equation $F_u(\lambda^*, u_{\lambda^*})\phi = 0$ has no nontrivial solution. From Lemma 3.8, $F_u$ maps $\mathbb{R} \times H^1(\mathbb{R}^N)$ onto $H^{-1}(\mathbb{R}^N)$. Applying the implicit function theorem to $F$, we can find a neighbourhood $(\lambda^* - \delta, \lambda^* + \delta)$ of $\lambda^*$ such that $(1.1)_{\lambda}$ possesses a solution $u_\lambda$ if $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$. This is contradictory to the definition of $\lambda^*$. Hence, we obtain that $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$.

Next, we are going to prove that $u_{\lambda^*}$ is unique. In fact, suppose $(1.1)_{\lambda^*}$ has another solution $U_{\lambda^*} \geq u_{\lambda^*}$. Set $w = U_{\lambda^*} - u_{\lambda^*}$; we have

$$0 = \int_{\mathbb{R}^N} \lambda^* K[(w + u_{\lambda^*})^p - u_{\lambda^*}^p] \phi_1 \, dx$$

By $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$, we have that the problem

$$-\Delta \phi + \phi = \lambda^* p K u_{\lambda^*}^{p-1} \phi, \quad \phi \in H^1(\mathbb{R}^N)$$

possesses a positive solution $\phi_1$.

Multiplying (3.13) by $\phi_1$ and (3.14) by $w$, integrating and subtracting we deduce that

$$0 = \frac{1}{2} p (p - 1) \int_{\mathbb{R}^N} \lambda^* K \xi_{\lambda^*}^{p-2} w^2 \phi_1 \, dx,$$

where $\xi_{\lambda^*} \in (u_{\lambda^*}, u_{\lambda^*} + w)$. Thus $w \equiv 0$.

4. **Existence of second solution**

The existence of a second solution of $(1.1)_{\lambda}$, $\lambda \in (0, \lambda^*)$, will be established via the mountain pass theorem. When $0 < \lambda < \lambda^*$, we have known that $(1.1)_{\lambda}$ has a minimal positive solution $u_\lambda$ by Lemma 3.2, then we need only to prove that $(1.1)_{\lambda}$ has another positive solution in the form of $U_\lambda = u_\lambda + v_\lambda$, where $v_\lambda$ is a solution of the following problem:

$$(4.1)_{\lambda} \left\{ \begin{array}{l}
-\Delta v + v = \lambda K[(v + u_\lambda)^p - u_\lambda^p] \text{ in } \mathbb{R}^N, \\
v \in H^1(\mathbb{R}^N), v > 0 \text{ in } \mathbb{R}^N,
\end{array} \right.$$

The corresponding variational functional of $(4.1)_{\lambda}$ is

$$J(\lambda)(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) - \lambda \int_{\mathbb{R}^N} \int_{0}^{v^+} K[(s + u_\lambda)^p - u_\lambda^p] \, ds \, dx, \quad v \in H^1(\mathbb{R}^N).$$
The following lemma comes from the fact

$$\lim_{s \to 0} \frac{(u_\lambda + s)^p - u_\lambda^p - pu_\lambda^{p-1}s}{s} = 0$$

and

$$\lim_{s \to \infty} \frac{(u_\lambda + s)^p - u_\lambda^p - pu_\lambda^{p-1}s}{s^p} = 1$$

**Lemma 4.1.** For any $\varepsilon > 0$, there is a positive constant $c_\varepsilon$ such that

$$(u_\lambda + s)^p - u_\lambda^p - pu_\lambda^{p-1}s \leq \varepsilon u_\lambda^{p-1}s + c_\varepsilon s^p$$

for all $s \geq 0$.

**Lemma 4.2.** Under condition $(k1)$, then there exist positive constants $\rho$ and $\alpha$, such that

$$J_\lambda(v) \geq \alpha > 0, \ v \in H^1(\mathbb{R}^N), \ ||v|| = \rho.$$ 

**Proof:** By Lemma 4.1, we have

$$J_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) \ dx - \frac{1}{2} \lambda p \int_{\mathbb{R}^N} K u_\lambda^{p-1}(v^+)^2 \ dx$$

$$- \lambda \int_{\mathbb{R}^N} \int_{0}^{v^+} K[(u_\lambda + s)^p - u_\lambda^p - pu_\lambda^{p-1}s] \ ds \ dx$$

$$(4.2) \geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) \ dx - \lambda p \int_{\mathbb{R}^N} K u_\lambda^{p-1}(v^+)^2 \ dx$$

$$- \lambda \int_{\mathbb{R}^N} K \left[ \frac{\varepsilon}{2} u_\lambda^{p-1}(v^+)^2 + c_\varepsilon (v^+)^{p+1} \right] \ dx.$$

Furthermore, from the definition $\sigma_\lambda(u_\lambda)$ in (3.2), we have

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) \ dx \geq \sigma_\lambda(u_\lambda)p \int_{\mathbb{R}^N} K u_\lambda^{p-1}(v^+)^2 \ dx,$$

and, therefore, by (4.2) we obtain

$$J_\lambda(v) \geq \frac{1}{2} \sigma_\lambda(u_\lambda)^{-1} (\sigma_\lambda(u_\lambda) - \lambda - \frac{\varepsilon}{2} \lambda) ||v||^2 - \lambda c(p + 1)^{-1} \int_{\mathbb{R}^N} K(v^+)^p \ dx.$$ 

Since $\sigma_\lambda(u_\lambda) > \lambda$, by property (ii) in Lemma 3.3, the boundedness of $K$, and the Sobolev inequality imply that for small $\varepsilon > 0$,

$$J_\lambda(v) \geq \frac{1}{4} \sigma_\lambda(u_\lambda)^{-1} (\sigma_\lambda(u_\lambda) - \lambda) ||v||^2 - \lambda c ||v||^{p+1},$$

and the conclusion in Lemma 4.2 follows.
We need the following concentration compactness principle to prove our result:

**Lemma 4.3.** Assume condition (k1) holds. Let \( \{v_k\} \) be a \((PS)_c\) sequence of \( J_\lambda \) in \( H^1(\mathbb{R}^N) \):

\[
J_\lambda(v_k) = c + o(1) \text{ as } k \to \infty, \\
J'_\lambda(v_k) = o(1) \text{ strong in } H^{-1}(\mathbb{R}^N).
\]

Then there exists a subsequence (still denoted by) \( \{v_k\} \) for which the following holds: there exist an integer \( l \geq 0 \), sequence \( \{x_k^i\} \subset \mathbb{R}^N \), a solution \( v_\lambda \) of (4.1)\(\lambda \) and solutions \( v_i^\lambda \) of (2.1)\(\lambda \), for \( 1 \leq i \leq l \), such that

\[
\begin{cases}
v_k \rightharpoonup v_\lambda \text{ weakly in } H^1(\mathbb{R}^N); \\
v_k - \left[v_\lambda + \sum_{i=1}^l \bar{v}_\lambda^i (\cdot - x_k^i)\right] \to 0 \text{ strongly in } H^1(\mathbb{R}^N); \\
J_\lambda(v_k) = J_\lambda(v_\lambda) + \sum_{i=1}^l I_\lambda^{\infty}(\bar{v}_\lambda^i) + o(1).
\end{cases}
\]

where we agree that in the case \( l = 0 \) the above holds without \( v_\lambda, x_k^i \).

**Proof:** Lemma 4.3 can be derived directly from the arguments in Bahri and Lions [3] (or [4, 14, 17]). We omit it.

**Lemma 4.4.** Assume condition (k1) holds, then

(i) there exists \( t_0 > 0 \), such that

\[ J_\lambda(t\omega_\lambda) < 0 \text{ for all } t \geq t_0. \]

(ii) the following inequality holds

\[ 0 < \sup_{t \geq 0} J_\lambda(t\omega_\lambda) < I_\lambda^{\infty}(\omega_\lambda) = M_\lambda^{\infty}. \]

**Proof:** By \( \omega_\lambda \) is the ground state solution of (2.1)\(\lambda \) and condition (k1), then we have

\[
J_\lambda(t\omega_\lambda) = \frac{1}{2} t^2 \int_{\mathbb{R}^N} (|\nabla \omega_\lambda|^2 + |\omega_\lambda|^2) \, dx - \frac{1}{p+1} t^{p+1} \int_{\mathbb{R}^N} \lambda K(x)\omega_\lambda^{p+1} \, dx \\
- \int_{\mathbb{R}^N} \int_{0}^{t\omega_\lambda} \lambda K(x) \left[(s + u_\lambda)^p - u_\lambda^p - s^p\right] ds \, dx
\]

\[
(4.3) \leq \frac{1}{2} t^2 \|\omega_\lambda\|^2 - \frac{1}{p+1} t^{p+1} \int_{\mathbb{R}^N} \lambda K_\infty \omega_\lambda^{p+1}(x) \, dx
\]

\[
\leq c_1 t^2 - c_2 t^{p+1}
\]

where \( c_1 = 1/2\|\omega_\lambda\|^2 \), \( c_2 = 1/(p+1) \int_{\mathbb{R}^N} \lambda K_\infty \omega_\lambda^{p+1}(x) \, dx \) are independent of \( t \). From (4.3), we conclude the result (i).
From (i), we easily see that the left hand of (ii) holds and we need only to show that the right hand of (ii) holds. By (i), we have that there exists \( t_2 > 0 \) such that
\[
\sup_{t \geq 0} J_\lambda(t\omega_\lambda) = \sup_{0 \leq t \leq t_2} J_\lambda(t\omega_\lambda).
\]
Since \( J \) is continuous in \( H^1(\mathbb{R}^N) \), there exists \( t_1 > 0 \) such that
\[
J_\lambda(t\omega_\lambda) < M^\infty_\lambda, \text{ for } 0 \leq t < t_1.
\]
Then, to prove (ii) we now only to prove the following inequality:
\[
\sup_{t_1 \leq t \leq t_2} J_\lambda(t\omega_\lambda) < I^\infty_\lambda(\omega_\lambda) = M^\infty_\lambda.
\]

By the definition of \( J_\lambda \), we get
\[
J_\lambda(t\omega_\lambda) = \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla \omega_\lambda|^2 + \omega_\lambda^2) \, dx - \frac{t^{p+1}}{p+1} \int_{\mathbb{R}^N} \lambda K_\infty \omega_\lambda^{p+1} \, dx
+ \frac{t^{p+1}}{p+1} \int_{\mathbb{R}^N} \lambda (K_\infty - K(x)) \omega_\lambda^{p+1} \, dx
- \int_{\mathbb{R}^N} \int_{0}^{t\omega_\lambda} \lambda K(x)[(s + u_\lambda)^p - u_\lambda^p - s^p] \, ds \, dx.
\]
Since \( \omega_\lambda \) is the ground state solution of (2.1) and \( \sup_{t \geq 0} I^\infty_\lambda(t\omega_\lambda) = I^\infty_\lambda(\omega_\lambda) \), then we have
\[
J_\lambda(t\omega_\lambda) \leq I^\infty_\lambda(\omega_\lambda) - \frac{t^{p+1}}{p+1} \int_{\mathbb{R}^N} \lambda (K_\infty - K(x)) \omega_\lambda^{p+1} \, dx
- \int_{\mathbb{R}^N} \int_{0}^{t\omega_\lambda} \lambda K(x)[(s + u_\lambda)^p - u_\lambda^p - s^p] \, ds \, dx.
\]
By condition (k1) and \((t_1 + t_2)^p \geq (\neq)t_1^p + t_2^p\) for all \( t_1 \geq 0, t_2 \geq 0, p > 1. \) Therefore, we obtain that
\[
J_\lambda(t\omega_\lambda) \leq I^\infty_\lambda(\omega_\lambda) - \inf_{t_1 \leq t \leq t_2} \int_{\mathbb{R}^N} \int_{0}^{t\omega_\lambda} \lambda K(x)[(s + u_\lambda)^p - u_\lambda^p - s^p] \, ds \, dx
< M^\infty_\lambda.
\]
Therefore (ii) holds.

**Proposition 4.5.** Suppose condition (k1) holds. Then problem (4.1) has at least one solution for \( \lambda \in (0, \lambda^*) \).

**Proof:** By Lemma 4.4 (i), we know that there is \( t_0 > 0 \) such that \( J_\lambda(t_0\omega_\lambda) < 0. \) We set
\[
\Gamma = \left\{ \gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \, \gamma(1) = t_0\omega_\lambda \right\},
\]
then, by Lemma 4.2 and Lemma 4.4 (ii) we get

\[(4.4) \quad 0 < c = \inf_{\gamma \in \Gamma} \max_{0 \leq s \leq 1} J_{\lambda}(\gamma(s)) < M^\infty_{\lambda}.
\]

Applying the mountain pass lemma of Ambrosetti and Rabinowitz [2], there exists a \((PS)_c\)-sequence \(\{v_k\}\) such that

\[J_{\lambda}(v_k) \to c \quad \text{and} \quad J'_{\lambda}(v_k) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N).
\]

By Lemma 4.3, there exist a subsequence, still denoted by \(\{v_k\}\), an integer \(l \geq 0\), a solution \(v_\lambda\) of \((4.1)_{\lambda}\) and solutions \(\overline{v}^i_\lambda\) of \((2.1)_{\lambda}\), for \(1 \leq i \leq l\), such that

\[(4.5) \quad c = J_{\lambda}(v_\lambda) + \sum_{i=1}^{l} I^\infty_{\lambda}(\overline{v}^i_\lambda).
\]

By the strong maximum principle, to complete the proof, we only need to prove \(v_\lambda \not\equiv 0\) in \(\mathbb{R}^N\). In fact, by (4.4) and (4.5), we have

\[c = J_{\lambda}(v_\lambda) \geq \alpha > 0 \quad \text{if} \quad l = 0, \quad M^\infty_{\lambda} > c \geq J_{\lambda}(v_\lambda) + M^\infty_{\lambda} \quad \text{if} \quad l \geq 1.
\]

This implies \(v_\lambda \not\equiv 0\) in \(\mathbb{R}^N\).

5. Properties and bifurcation of solutions

In this section, we shall give some further properties and bifurcation of solutions for problem \((1.1)_{\lambda}\). Now, we set

\[A = \{(\lambda, u) : u \text{ satisfies } (1.1)_{\lambda}, \lambda \in [0, \lambda^*]\}.
\]

For each \((\lambda, u) \in A\), let \(\sigma_\lambda(u)\) denote the number defined by (3.2), which is the smallest eigenvalue of the problem (3.3).

We always assume that condition \((h1)\) and \((k1)\) hold. By Lemma 3.6, we have \(A \subset L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)\). Moreover, if we assume that,

\[h(x), K(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N),
\]

then by elliptic regular theory ([10]), we can deduce that \(A \subset C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)\).

**Lemma 5.1.** Let \(u\) be a solution and \(u_\lambda\) be the minimal solution of \((1.1)_{\lambda}\) for \(\lambda \in (0, \lambda^*)\). Then

(i) \(\sigma_\lambda(u) > \lambda\) if and only if \(u = u_\lambda\);

(ii) \(\sigma_\lambda(U_\lambda) < \lambda\), where \(U_\lambda\) is the second solution of \((1.1)_{\lambda}\) constructed in Section 4.
Proof: Now, let $\psi \geq 0$ and $\psi \in H^1(\mathbb{R}^N)$. Since $u$ and $u_\lambda$ are the solution of (1.1)$_\lambda$, then

$$
\int_{\mathbb{R}^N} \nabla \psi \cdot \nabla (u_\lambda - u) \, dx + \int_{\mathbb{R}^N} \psi (u_\lambda - u) \, dx = \lambda \int_{\mathbb{R}^N} K(u_\lambda^p - u^p) \psi \, dx
$$

(5.1)

$$
= \lambda \int_{\mathbb{R}^N} \left( \int_u^{u_\lambda} t^{p-1} \, dt \right) pK \psi \, dx

\geq \lambda \int_{\mathbb{R}^N} pK u^{p-1}(u_\lambda - u) \psi \, dx.
$$

Let $\psi = (u - u_\lambda)^+ \geq 0$ and $\psi \in H^1(\mathbb{R}^N)$. If $\psi \neq 0$, then (5.1) implies

$$
-\int_{\mathbb{R}^N} (|\nabla \psi|^2 + \psi^2) \, dx \geq -\lambda \int_{\mathbb{R}^N} pK u^{p-1} \psi^2 \, dx
$$

and, therefore, the definition of $\sigma_\lambda(u)$ implies

$$
\int_{\mathbb{R}^N} (|\nabla \psi|^2 + \psi^2) \, dx \leq \lambda \int_{\mathbb{R}^N} pK u^{p-1} \psi^2 \, dx

< \sigma_\lambda(u) \int_{\mathbb{R}^N} pK u^{p-1} \psi^2 \, dx

\leq \int_{\mathbb{R}^N} (|\nabla \psi|^2 + \psi^2) \, dx,
$$

which is impossible. Hence $\psi \equiv 0$, and $u = u_\lambda$ in $\mathbb{R}^N$. On the other hand, by Lemma 3.3, we also have that $\sigma_\lambda(u_\lambda) > \lambda$. This completes the proof of (i).

By (i), we get that $\sigma_\lambda(U_\lambda) \leq \lambda$ for $\lambda \in (0, \lambda^*)$. We claim that $\sigma_\lambda(U_\lambda) = \lambda$ cannot occur. We proceed by contradiction. Set $w = U_\lambda - u_\lambda$; we have

$$
-\Delta w + w = \lambda K \left[ U_\lambda^p - (U_\lambda - w)^p \right], \ w > 0 \text{ in } \mathbb{R}^N.
$$

(5.2)

By $\sigma_\lambda(U_\lambda) = \lambda$, we have that the problem

$$
-\Delta \phi + \phi = \lambda pK U_\lambda^{p-1} \phi, \ \phi \in H^1(\mathbb{R}^N)
$$

(5.3)

possesses a positive solution $\phi_1$.

Multiplying (5.2) by $\phi_1$ and (5.3) by $w$, integrating and subtracting we deduce that

$$
0 = \int_{\mathbb{R}^N} \lambda K \left[ U_\lambda^p - (U_\lambda - w)^p - pU_\lambda^{p-1}w \right] \phi_1 \, dx

= -\frac{1}{2} p(p-1) \int_{\mathbb{R}^N} \lambda K \xi_\lambda^{p-2} w^2 \phi_1 \, dx,
$$

where $\xi_\lambda \in (u_\lambda, U_\lambda)$. Thus $w \equiv 0$, that is $U_\lambda = u_\lambda$ for $\lambda \in (0, \lambda^*)$. This is a contradiction. Hence, we have that $\sigma_\lambda(U_\lambda) < \lambda$ for $\lambda \in (0, \lambda^*)$.

Remark 5.2. Since $\sigma_\lambda(U_\lambda) < \lambda$, one may employ a similar argument to the used for $u_\lambda$ to show that $U_\lambda$ is strictly decreasing in $\lambda$, $\lambda \in (0, \lambda^*)$. 
Proposition 5.3. Let $u_\lambda$ be the minimal solution of (1.1)$_\lambda$. Then $u_\lambda$ is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for all $\lambda \in [0, \lambda^*]$ and

$$u_\lambda \to u_0 \text{ in } L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \text{ as } \lambda \to 0.$$ 

where $u_0$ is the unique positive solution of (1.1)$_0$.

**Proof:** By Lemma 3.2, 3.3, and 3.7, we can deduce $\|u_\lambda\|_\infty \leq \|u_{\lambda^*}\|_\infty \leq c$, for $\lambda \in [0, \lambda^*]$. By (3.8), we have that

$$\|u_\lambda\| \leq \frac{p}{p-1} \|h\|_{H^{-1}}.$$ 

Hence, $u_\lambda$ is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for $\lambda \in [0, \lambda^*]$.

Now, let $w_\lambda = u_\lambda - u_0$, then $w_\lambda$ satisfies the following equation

$$(5.4)_\lambda$$

$$-\Delta w_\lambda + w_\lambda = \lambda K u^p_\lambda \text{ in } \mathbb{R}^N,$$

and by $u_\lambda$ is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, we have that

$$\|w_\lambda\|^2 = \int_{\mathbb{R}^N} \lambda K u^p_\lambda w_\lambda \, dx 
\leq \lambda \|K\|_\infty \|u_\lambda\|_\infty^{p-1} \|u_\lambda\|_2 \|w_\lambda\|_2 
\leq c\lambda,$$

where $c$ is independent of $\lambda$. Hence, we obtain that $u_\lambda \to u_0$ in $H^1(\mathbb{R}^N)$ as $\lambda \to 0$.

Now, let $q_0 = N/2 + 2 > \max\{N/2, 2\}$ and by $u_\lambda$ is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, then we have that $\lambda K u^p_\lambda \in L^{q_0}(\mathbb{R}^N)$. By Lemma 3.6 and using (5.4)$_\lambda$, we have

$$w_\lambda \in W^{2,2}(\mathbb{R}^N) \cap W^{2,q_0}(\mathbb{R}^N).$$

By the Sobolev embedding theorem, Lemma 3.6 and $u_{\lambda^*} \geq u_\lambda > 0$ for $\lambda \in [0, \lambda^*]$, we have that

$$\|w_\lambda\|_\infty \leq c_1 \|w_\lambda\|_{W^{2,q_0}(\mathbb{R}^N)} 
\leq c_2 (\|\lambda K u^p_\lambda\|_{q_0} + \|w_\lambda\|_{q_0}) 
\leq c_3 (\lambda \|u^p_\lambda\|_{q_0} + \|w_\lambda\|_\infty^{(q_0-2)/(q_0)} \|w_\lambda\|_2^{2/(q_0)}) 
\leq c (\lambda + \lambda^{1/(q_0)})$$

where $c$ is independent of $\lambda$. Hence, we obtain that $u_\lambda \to u_0$ in $L^\infty(\mathbb{R}^N)$ as $\lambda \to 0$. 

Proposition 5.4. For $\lambda \in (0, \lambda^*)$, let $U_\lambda$ be the positive solution of (1.1)$_\lambda$ with $U_\lambda > u_\lambda$, then $U_\lambda$ is unbounded in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, that is

$$\lim_{\lambda \to 0} \|U_\lambda\| = \lim_{\lambda \to 0} \|U_\lambda\|_\infty = \infty.$$
**Proof:** Firstly, we show that \( \{ U_\lambda : \lambda \in (0, \lambda^*) \} \) is unbounded in \( H^1(\mathbb{R}^N) \). Since \( U_\lambda = u_\lambda + v_\lambda \), we only need to show that \( \{ v_\lambda : \lambda > 0 \} \) is unbounded in \( H^1(\mathbb{R}^N) \). If not, then

\[
(5.5) \quad \| v_\lambda \| \leq M
\]

for all \( \lambda \in (0, \lambda^*) \). It is easily to see that for any \( \delta > 0 \), \( \{ U_\lambda \}_{\lambda \geq \delta} \) is bounded in \( H^1(\mathbb{R}^N) \), we may assume \( \lambda \in (0, \delta] \).

Choose \( \lambda_n \downarrow 0 \) and let \( v_{\lambda_n} \) be the corresponding solutions constructed by Proposition 4.5. By the Hölder inequality and the Sobolev embedding theorem, we obtain that

\[
\int_{\mathbb{R}^N} (|\nabla v_{\lambda_n}|^2 + |v_{\lambda_n}|^2) \, dx = \int_{\mathbb{R}^N} \lambda_n K [U_{\lambda_n}^p - u_{\lambda_n}^p] v_{\lambda_n} \, dx \\
\leq c \lambda_n \| U_{\lambda_n}^p \|_{p+1} \| v_{\lambda_n} \|_{p+1} \\
\leq c \lambda_n \| U_{\lambda_n} \|^p \| v_{\lambda_n} \| \\
\leq c_1 \lambda_n
\]

for some constant \( c_1 \), independent of \( v_{\lambda_n} \), where we have used (5.5) and the boundedness of \( \{ u_{\lambda_n} \} \) in \( H^1(\mathbb{R}^N) \). Hence, we have \( \lim_{n \to \infty} \| v_{\lambda_n} \|^2 = 0 \). It implies that

\[
(5.6) \quad \lim_{n \to \infty} \| v_{\lambda_n} \|_2 = 0.
\]

On the other hand, we notice that \( U_{\lambda} = u_{\lambda} + v_{\lambda} \) is decreasing and \( u_{\lambda} \) is increasing in \( \lambda \). Therefore, \( v_{\lambda} \) is decreasing in \( \lambda \), which implies

\[
v_{\lambda_n} \geq v_{\delta} \quad \text{for all } n,
\]

then we obtain that

\[
\| v_{\lambda_n} \|_2 \geq \| v_{\delta} \|_2 > 0 \quad \text{for all } n.
\]

which contradicts (5.6). This implies that \( \{ U_{\lambda} : \lambda \in (0, \lambda^*) \} \) is unbounded in \( H^1(\mathbb{R}^N) \).

Now, we show that \( \{ U_{\lambda} : \lambda \in (0, \lambda^*) \} \) is unbounded in \( L^\infty(\mathbb{R}^N) \). We proceed by contradiction. Assume to the contrary that there exists \( c_0 > 0 \) such that

\[
\| U_{\lambda} \|_\infty \leq c_0 < \infty \quad \text{for all } \lambda \in (0, \lambda^*).
\]

Since \( U_{\lambda} \) is a solution of (1.1)\( \lambda \), we have that

\[
\| U_{\lambda} \|^2 = \int_{\mathbb{R}^N} \lambda K^{p+1} \, dx + \int_{\mathbb{R}^N} hU_{\lambda} \, dx \\
\leq \lambda^{p-1}_0 \| K \|_\infty \| U_{\lambda} \|^2 + \| h \|_2 \| U_{\lambda} \|_2 \\
\leq c_1 \lambda \| U_{\lambda} \|^2 + c_2 \| U_{\lambda} \|,
\]

where \( c_1 \) and \( c_2 \) are independent of \( \lambda \). If we choose

\[
\lambda_0 = \min \left\{ \lambda^*, \frac{1}{2c_1} \right\},
\]
then there exists $c > 0$, independent of $\lambda$, such that $\|U_\lambda\| \leq c$ for all $\lambda \leq \lambda_0$. This is a contradiction to that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $H^1(\mathbb{R}^N)$. This completes the proof of Proposition 5.4.

In order to get bifurcation results we need the following Bifurcation Theorem which can be found in Crandall and Rabinowitz [7].

**Theorem A.** Let $X, Y$ be Banach space. Let $(\bar{\lambda}, \bar{x}) \in \mathbb{R} \times X$ and let $F$ be a continuously differentiable mapping of an open neighbourhood of $(\bar{\lambda}, \bar{x})$ into $Y$. Let the null-space

$$N(F_x(\bar{\lambda}, \bar{x})) = \text{span}\{x_0\}$$

be one-dimensional and $\text{codim} \ R(F_x(\bar{\lambda}, \bar{x})) = 1$. Let $F_\lambda(\bar{\lambda}, \bar{x}) \not\in R(F_x(\bar{\lambda}, \bar{x}))$. If $Z$ is the complement of $\text{span}\{x_0\}$ in $X$, then the solutions of $F(\lambda, x) = F(\bar{\lambda}, \bar{x})$ near $(\bar{\lambda}, \bar{x})$ form a curve

$$(\lambda(s), x(s)) = (\bar{\lambda} + \tau(s), \bar{x} + sx_0 + z(s)),$$

where

$$s \to (\tau(s), z(s)) \in \mathbb{R} \times Z$$

is continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.

**Proof of Theorem 1.1 and Theorem 1.2:** Theorem 1.1 now follows from Lemma 3.2, 3.3, 3.4, 3.10, 5.1 and Proposition 4.5. The conclusion (i) and (ii) of Theorem 1.2 follow immediately from Lemma 3.2, Remark 5.2 and Proposition 5.3, 5.4. Now we are going to prove that $(\lambda^*, u_{\lambda^*})$ is a bifurcation point in $C^2,\alpha(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ by using an idea in [12]. We also assume that $K(x)$ and $h(x)$ are in $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and define

$$F : \mathbb{R} \times C^2,\alpha(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \to C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$$

by

$$F(\lambda, u) = \Delta u - u + \lambda K(u^+)^p + h(x).$$

where $C^2,\alpha(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ and $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ are endowed with the natural norm; then they become Banach spaces. It can be proved easily that $F(\lambda, u)$ is differentiable. From Lemma 3.8 and Remark 3.9, we know that

$$F_u(\lambda, u)w = \Delta w - w + \lambda pK u^{p-1}_x w$$

is an isomorphism of $\mathbb{R} \times C^2,\alpha(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ onto $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. It follows from Implicit Function Theorem that the solutions of $F(\lambda, u) = 0$ near $(\lambda, u_\lambda)$ are given by a continuous curve.

Now we are going to prove that $(\lambda^*, u_{\lambda^*})$ is a bifurcation point of $F$. We show first that at the critical point $(\lambda^*, u_{\lambda^*})$, Theorem A applies. Indeed, from Lemma 3.10,
problem (3.14) has a solution \( \phi_1 > 0 \) in \( \mathbb{R}^N \). By the standard elliptic regular theory, we have that \( \phi_1 \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \) if \( h \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \). Thus

\[
F_u(\lambda^*, u_{\lambda^*})\phi = 0, \quad \phi \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)
\]

has a solution \( \phi_1 > 0 \). This implies that \( N(F_u(\lambda^*, u_{\lambda^*})) = \text{span}\{\phi_1\} = 1 \) is one-dimensional and codim \( R(F_u(\lambda^*, u_{\lambda^*})) = 1 \) by the Fredholm alternative. It remains to check that \( F_\lambda(\lambda^*, u_{\lambda^*}) \not\in R(F_u(\lambda^*, u_{\lambda^*})) \).

Assuming the contrary would imply existence of \( v \neq 0 \) such that

\[
\Delta v - v + \lambda^* pKu_{\lambda^*}^{p-1}v = K u_{\lambda^*}^p, \quad v \in H^1(\mathbb{R}^N).
\]

From \( F_u(\lambda^*, u_{\lambda^*})\phi_1 = 0 \), we conclude that \( \int_{\mathbb{R}^N} Ku_{\lambda^*}^p \phi_1 \, dx = 0 \). This is impossible because \( K(x) \geq 0, K(x) \not\equiv 0, u_{\lambda^*}(x) > 0 \) and \( \phi_1(x) > 0 \) in \( \mathbb{R}^N \).

Applying Theorem A, we conclude that \( (\lambda^*, u_{\lambda^*}) \) is a bifurcation point near which, the solution of (1.1) form a curve \((\lambda^* + \tau(s), u_{\lambda^*} + s\phi_1 + z(s))\) with \( s \) near \( s = 0 \) and \( \tau(0) = \tau'(0) = 0, z(0) = z'(0) = 0 \). We claim that \( \tau''(0) < 0 \) which implies that the bifurcation curve turns strictly to the left in \((\lambda, u)\) plane.

Since \( u_{\lambda^*}(x) \to 0 \) as \(|x| \to \infty\), we have, for \(|x| \) large,

\[
0 = \Delta \phi_1 - \phi_1 + \lambda^* pKu_{\lambda^*}^{p-1} \phi_1 = \Delta \phi_1 - \frac{1}{4} \phi_1.
\]

It is well-known that the equation \( \Delta w - w/4 = -w^p \) in \( \mathbb{R}^N \) has a unique positive radial symmetric solution, denoted by \( \bar{w} \) (see Bahri and Lions [3] and the references there), and there exists \( c_1 > 0 \) such that

\[
\bar{w}(|x|) e^{|x|/2} |x|^{(N-1)/2} \to c_1.
\]

Since \( \Delta \bar{w} - \bar{w}/4 = -\bar{w}^p \leq 0 \) in \( \mathbb{R}^N \), hence we obtain by the maximum principle that

\[
(5.7) \quad \phi_1(x) \leq c_2 e^{-|x|/2} |x|^{-(N-1)/2} \quad \text{for } |x| \text{ large,}
\]

for some \( c_2 > 0 \).

From (3.10) and (5.7) and the Holder’s inequality, we derive that

\[
(5.8) \quad \int_{\mathbb{R}^N} Ku_{\lambda^*}^{p-2} \phi_1^3 \, dx \leq c \int_{\mathbb{R}^N} Ku_{\lambda^*}^{p-1} \phi_1 \, dx
\]

\[
\leq c \left( \int_{\mathbb{R}^N} u_{\lambda^*}^{p+1} \, dx \right)^{(p-1)/(p+1)} \left( \int_{\mathbb{R}^N} e^{-(p+1)/4 |x|} \, dx \right)^{2/(p+1)} < \infty.
\]

Since \( \lambda = \lambda^* + \tau(s) \), \( u = u_{\lambda^*} + s\phi_1 + z(s) \) in

\[
(5.9) \quad -\Delta u + u - \lambda K u^p - h = 0, \quad u > 0, \quad u \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N).
\]
Differentiating (5.9) in s twice, we have
\[-\Delta u_{ss} + u_{ss} - \lambda pK u^{p-1}_{ss} u_s - 2\lambda_s pK u^{p-1} u_s - \lambda p(p-1) K u^{p-2} u_s^2 - \lambda_{ss} K u^p = 0.\]

Setting here s = 0 and using the facts that \(\tau'(0) = 0\), \(u_s = \phi_1(x)\) and \(u = u_{\lambda^*}\) as \(s = 0\), we obtain
\[-\Delta u_{ss} + u_{ss} - \lambda^* pK u^{p-1}_{\lambda^*} u_s - \lambda^* p(p-1) K u^{p-2}_{\lambda^*} \phi_1^2 - \tau''(0) K u^p_{\lambda^*} = 0\]

Multiplying \(F_u(\lambda^*, u_{\lambda^*}) \phi_1 = 0\) by \(u_{ss}\), and (5.10) by \(\phi_1\), integrating and subtracting the result, and by (5.8) we obtain
\[
\int_{\mathbb{R}^N} \lambda^* p(p-1) K u^{p-2}_{\lambda^*} \phi_1^3 \, dx + \tau''(0) \int_{\mathbb{R}^N} K u^p_{\lambda^*} \phi_1 \, dx = 0,
\]
which immediately gives \(\tau''(0) < 0\). Thus
\[
\begin{align*}
    u_\lambda &\to u_{\lambda^*} \quad \text{in} \quad C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \quad \text{as} \quad \lambda \to \lambda^*, \\
    U_\lambda &\to u_{\lambda^*} \quad \text{in} \quad C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \quad \text{as} \quad \lambda \to \lambda^*.
\end{align*}
\]

Using Lemma 3.8, Remark 3.9, the Implicit Function Theorem and the uniqueness of the positive ground-state solution of (1.1)0, we can easily prove that
\[
u_\lambda \to u_0 \quad \text{in} \quad C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \quad \text{as} \quad \lambda \to 0,
\]
which proves Theorem 1.2. \(\square\)

REFERENCES


