On the monotonicity properties of additive representation functions.

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Volume 72 Number 1 August 2005
ON THE MONOTONICITY PROPERTIES OF ADDITIVE REPRESENTATION FUNCTIONS

YONG-GAO CHEN, ANDRÁS SÁRKÖZY, VERA T. SÓS AND MIN TANG

If \( A \) is a set of positive integers, let \( R_1(n) \) be the number of solutions of \( a + a' = n, \ a, \ a' \in A \), and let \( R_2(n) \) and \( R_3(n) \) denote the number of solutions with the additional restrictions \( a < a' \), and \( a \leq a' \) respectively. The monotonicity properties of the three functions \( R_1(n) \), \( R_2(n) \), and \( R_3(n) \) are studied and compared.

1. Introduction

Let \( \mathbb{N} \) denote the set of positive integers, let \( \mathcal{A} \subset \mathbb{N} \) be an infinite set, and put \( A(n) = |\{a : a \leq n, \ a \in \mathcal{A}\}| \). For \( n = 0, 1, 2, \ldots \), let

\[
R_1(n) = R_1(\mathcal{A}, n), \quad R_2(n) = R_2(\mathcal{A}, n), \quad R_3(n) = R_3(\mathcal{A}, n)
\]

denote the number of solutions of

\[
a + a' = n, \quad a, \ a' \in \mathcal{A},
\]
\[
a + a' = n, \quad a, \ a' \in \mathcal{A}, \quad a < a'
\]
\[
a + a' = n, \quad a, \ a' \in \mathcal{A}, \quad a \leq a',
\]
respectively.

Erdős, Sárközy and Sós [3, 4] and Balasubramanian [2] studied the monotonicity properties of the functions \( R_1(n) \), \( R_2(n) \) and \( R_3(n) \). Somewhat unexpectedly, it turned out that the monotonicity properties of the three representation functions differ significantly. In particular, Erdős, Sárközy and Sós proved in [3] that \( R_1(n) \) can be monotonically increasing from a certain point on only in the trivial way:

Theorem A. The function \( R_1(n) \) is eventually increasing; that is, there exists an integer \( n_0 \) with

\[
R_1(n + 1) \geq R_1(n) \quad \text{for } n \geq n_0
\]
If and only if $\mathbb{N} \setminus A$ is finite; that is, there exists an integer $n_1$ with

$$A \cap \{n_1, n_1 + 1, n_1 + 2, \ldots\} = \{n_1, n_1 + 1, n_1 + 2, \ldots\}$$

In [3] the following was also proved.

**Theorem B.** If $A \subset \mathbb{N}$ is an infinite set such that

$$(1) \quad A(n) = o\left(\frac{n}{\log n}\right),$$

then the function $R_2(n)$ cannot be eventually increasing.

In [3] they also claimed the following result:

**Theorem C.** Let $B$ be a set of positive integers such that

(i) $B$ is a "Sidon set", that is,

$$b_1 + b_2 = b_3 + b_4, \quad b_1, b_2, b_3, b_4 \in B, \quad b_1 \leq b_2, b_3 \leq b_4$$

imply that $b_1 = b_3$ and $b_2 = b_4$,

(ii) all the elements of $B$ are even, and

(iii) $b, b' \in B$ implies that $(b + b')/2 \notin B$.

Then the complement of $B$, that is, the set

$$(2) \quad \mathcal{A} = \mathbb{N} \setminus B$$

is such that the function $R_2(n) = R_2(A, n)$ is monotonically increasing.

However, this theorem is false in its original form stated above: it is easy to check that the set $B = \{2, 2^2, \ldots, 2^n, \ldots\}$ satisfies conditions (i), (ii) and (iii) in the theorem; but defining $\mathcal{A}$ by (2), we have

$$R_2(\mathcal{A}, 2^n) = 2^{n-1} - n + 1$$

and

$$R_2(\mathcal{A}, 2^n + 1) = 2^{n-1} - n$$

so that

$$R_2(\mathcal{A}, 2^n) > R_2(\mathcal{A}, 2^n + 1)$$

and thus $R_2(A, n)$ is not eventually increasing. The error in the theorem is due to the fact that a computational error was made in the last line of (28) in [3] and thus the formula stated there is wrong.

In [4] Erdős, Sárközy and Sós proved:

**Theorem D.** If $A \subset \mathbb{N}$ is an infinite set such that

$$(3) \quad \lim_{n \to +\infty} \frac{n - A(n)}{\log n} = +\infty,$$
then we have

\[ \limsup_{N \to +\infty} \sum_{k=1}^{N} (R_3(2k) - R_3(2k + 1)) = +\infty. \]

It was also shown in [4] that this result is near the best possible:

**Theorem E.** There exists an infinite sequence \( \mathcal{A} \subset \mathbb{N} \) such that there are \( c(>0) \), \( n_0 \) so that

\[ n - A(n) > c \log n \quad (\text{for } n > n_0) \]

and

\[ \limsup_{N \to +\infty} \sum_{k=1}^{N} (R_3(2k) - R_3(2k + 1)) < +\infty. \]

Indeed, they proved this by showing that the set

\[ \mathcal{A} = \mathbb{N} \setminus \{17, 64, \ldots, 4^{2k} + 1, 4^{2k+1}, \ldots\} \]

satisfies (5) and (6)

In [6], Tang and Chen generalised Theorem D and gave a quantitative form of it. As a corollary, we have

**Theorem F.** If \( \mathcal{A} \subset \mathbb{N} \) is an infinite set such that

\[ \limsup_{n \to +\infty} \frac{n - A(n)}{\log n} = +\infty, \]

then we have

\[ \limsup_{N \to +\infty} \sum_{k=1}^{N} (R_3(2k) - R_3(2k + 1)) = +\infty. \]

(9) implies that \( R_3(2k) > R_3(2k + 1) \) infinitely often, thus it follows from Theorem F that:

**Theorem G.** If \( \mathcal{A} \subset \mathbb{N} \) is an infinite set such that (8) holds, then the function \( R_3(n) \) cannot be eventually increasing, that is, there is no \( n_0 \in \mathbb{N} \) with

\[ R_3(n + 1) \geq R_3(n) \quad \text{for } n \geq n_0. \]

Theorem G with (8) replaced by (3) has also been proved simultaneously and independently by Balasubramanian [2]. However, the following problem has not been solved yet (see [5, Problem 4]).
PROBLEM 1. Does there exist an infinite set $\mathcal{A} \subset \mathbb{N}$ such that $\mathbb{N} \setminus \mathcal{A}$ is infinite and $R_3(n)$ is eventually increasing?

By Theorem E, the set $\mathcal{A}$ in (7) seems to be a good candidate for being a set possessing the properties described in Problem 1, thus one might like to study the monotonicity of $R_3(\mathcal{A}, n)$ for this set $\mathcal{A}$. But for this set and $l \geq 2$, we have

$$R_3(\mathcal{A}, 4^{2l+4^{2l-2}} + 2) = R_3(\mathcal{A}, 4^{2l+4^{2l-2}} + 3) + 1.$$  

So the function $R_3(\mathcal{A}, n)$ cannot be eventually increasing.

Although Theorem F is near the best possible by Theorem E, this is not so with Theorem G which is the consequence of Theorem F, and perhaps Theorem G could be improved upon. It is even possible that the answer to the question in Problem 1 is negative; that is, $R_3(n)$ can be increasing from a certain point on only in the trivial way.

In this paper our goal is twofold. First we shall show that Theorem C can be corrected by slightly modifying it. The statement of Theorem C is true if we replace condition (iii) by

(iii)$'$ \hspace{1em} $b, b' \in \mathcal{B}$ implies that $(b + b') \notin \mathcal{B}$.

Indeed, we shall prove slightly more:

**Theorem 1.** Let $\mathcal{B} \subset \mathbb{N}$ be an infinite set all whose elements are even, and write $\mathcal{A} = \mathbb{N} \setminus \mathcal{B}$. Then $R_2(n) = R_2(\mathcal{A}, n)$ is eventually increasing, that is, there exists an integer $n_0$ with

$$R_2(n+1) \geq R_2(n) \quad \text{for} \ n \geq n_0,$$

if and only if

(i) \hspace{1em} $R_3(\mathcal{B}, n) \leq 1$ for $n \geq n_0$ and

(ii) \hspace{1em} $b, b' \in \mathcal{B}, b + b' \geq n_0$ imply that $(b + b') \notin \mathcal{B}$.

We remark that it can be shown easily by the greedy algorithm that there is an infinite set $\mathcal{B} \subset \{2, 4, 6, \ldots\}$ such that it satisfies (i) and (ii) in Theorem 1 and we have

$$B(n) = |\mathcal{B} \cap [0, n]| \gg n^{1/3}$$

(and by using a result of Ajtai, Komlós and Szemerédi [1], with a little work this lower bound could be improved to $\gg (n \log n)^{1/3}$). Then the complement $\mathcal{A} = \mathbb{N} \setminus \mathcal{B}$ of $\mathcal{B}$ satisfies

$$A(n) = |\mathcal{A} \cap [0, n]| = n - B(n) < n - cn^{1/3} \quad \text{(for large } n).$$

Thus by Theorem 1 it follows:

**Corollary 1.** There is an infinite set $\mathcal{A} \subset \mathbb{N}$ and $c > 0, n_0, n_1$ such that

$$A(n) < n - cn^{1/3} \quad \text{for } n \geq n_0$$
and $R_2(\mathcal{A}, n)$ is monotonically increasing for $n \geq n_1$.

We remark that there is a big gap between the lower and upper bounds given for $A(n)$ in (1) and (11). Unfortunately, we have not been able to tighten this gap and, in particular, we have not been able to answer the following question.

**Problem 2.** Is it true that if $\mathcal{A} \subseteq \mathbb{N}$ is an infinite set such that $R_2(n)$ is monotonically increasing from a certain point on, then we must have

$$\limsup_{n \to +\infty} \frac{A(n)}{n} = 1$$

or, perhaps, even

$$\lim_{n \to +\infty} \frac{A(n)}{n} = 1?$$

In the second half of this paper we shall prove a further partial result on $R_3(n)$ which seems to indicate that, perhaps, the answer to the question in **Problem 1** is negative, that is, $R_3(n)$ can be monotonically increasing only in the trivial way. We show if $\mathcal{A}$ is infinite and $R_3(n)$ is eventually increasing, then writing $\mathcal{B} = \{b_1 < b_2 < \cdots\} = \mathbb{N}\setminus\mathcal{A}$, by **Theorem G** there is a $C(= C(B)) > 1$ so that

$$b_n > C^n$$

for all large $n$. Now we shall show that if the elements of $\mathcal{B}$ grow quickly, then again $R_3(n)$ cannot be eventually increasing:

**Theorem 2.** Assume that $\mathcal{B} = \{b_1 < b_2 < \cdots\} \subseteq \mathbb{N}$ is an infinite sequence and define $\mathcal{A}$ by $\mathcal{A} = \mathbb{N}\setminus\mathcal{B}$. If

$$(12) \quad \lim_{n \to +\infty} (b_{n+1} - b_n) = +\infty,$$

then the function $R_3(n) = R_3(\mathcal{A}, n)$ is not eventually increasing; that is, there is no $n_0$ with

$$(13) \quad R_3(n + 1) \geq R_3(n) \quad \text{for } n \geq n_0.$$

We could prove other similar sufficient criteria. For example, we can prove that if all sufficiently large $b \in \mathcal{B}$ have the same parity, then $R_3(n)$ is not eventually increasing. However, we have not been able to settle **Problem 1**.

The results above reflect a striking and quite unexpected contrast between the monotonicity properties of the three representation functions: while $R_1(n)$ can be monotonically increasing only in the trivial way, by **Theorem 1** there are many sets $\mathcal{A}$ satisfying (11) so that $R_2(n)$ is monotonically increasing. Finally, $R_3(n)$ is closer to $R_1(n)$, than to $R_2(n)$: either it is monotonically increasing only in the trivial way or if there is a non-trivial $\mathcal{A}$ with this property then it must be such that it can be obtained from $\mathbb{N}$ by dropping only $< c \log n$ integers up to $n$ (for infinitely many $n$).
2. Proof of Theorem 1

Write

\[ B(n) = \left| \{ b : b \leq n, b \in \mathcal{B} \} \right|, \]
\[ \eta(i) = \begin{cases} 1 & \text{if } i \in \mathcal{B} \\ 0 & \text{if } i \notin \mathcal{B} \end{cases} \]

and

\[ \overline{R}(n) = R_3(\mathcal{B}, n) = \left| \{ (b, b') : b, b' \in \mathcal{B}, b \leq b', b + b' = n \} \right|. \]

Then

\[ R_2(n) = \left| \{(a, a') : a, a' \in \mathcal{A}, a < a', a + a' = n \} \right| = \sum_{1 \leq i < n/2} (1 - \eta(i))(1 - \eta(n-i)) = \sum_{1 \leq i < n/2} 1 - \left| \{ i : 1 \leq i \leq n-1, i \in \mathcal{B} \} \right| + \left| \{ (b, b') : b, b' \in \mathcal{B}, b \leq b', b + b' = n \} \right| = \sum_{1 \leq i < n/2} 1 - B(n-1) + \overline{R}(n). \]

Since the elements of \( \mathcal{B} \) are even, thus it follows that

\[ R_2(2k) = (k - 1) - B(2k - 2) + \overline{R}(2k) \]

and

\[ R_2(2k + 1) = k - B(2k) \]

then

\[ R_2(2k + 1) - R_2(2k) = 1 - (B(2k) - B(2k - 2)) - \overline{R}(2k) = 1 - \eta(2k) - \overline{R}(2k) \]

(14)

and

\[ R_2(2k) - R_2(2k - 1) = \overline{R}(2k). \]

The latter is always non-negative, thus (10) holds if and only if (14) is non-negative for \( 2k \geq n_0 \):

\[ 1 - \eta(2k) - \overline{R}(2k) \geq 0 \quad \text{(for } 2k \geq n_0). \]

Assume first that (10) holds. Since \( \eta(k) \geq 0 \), it follows from (15) that

\[ \overline{R}(2k) = R_3(\mathcal{B}, 2k) \leq 1 \quad \text{for } 2k \geq n_0. \]

(16)
The elements of \( \mathcal{B} \) are even, thus

\begin{equation}
R_3(\mathcal{B}, 2k + 1) = 0 \quad \text{for all } k \in \mathbb{N}.
\end{equation}

(i) in the theorem follows from (16) and (17). Moreover, if \( b, b' \in \mathcal{B} \) and \( b + b' \geq n_0 \), then writing \( b + b' = 2k \), we have \( R_3(\mathcal{B}, 2k) = \overline{R}(2k) \geq 1 \), thus it follows from (15) that \( \eta(2k) = \eta(b + b') = 0 \) so that \( b + b' \notin \mathcal{B} \) which proves (ii) in the theorem.

Assume now that (i) and (ii) in the theorem hold. If \( 2k \geq n_0 \), then by (i) we have \( \overline{R}(2k) = R_3(\mathcal{B}, 2k) \leq 1 \) so that \( \overline{R}(2k) = 0 \) or \( 1 \). If \( \overline{R}(2k) = 0 \), then by (i) \( \eta(2k) = 0 \) which proves (ii) in the theorem.

3. Proof of Theorem 2

We shall use proof by contradiction: assume that \( \mathcal{B} \subset \mathbb{N} \) satisfies (12), however, (13) holds for some \( n_0 \).

Define \( B(n), \eta(i) \) and \( \overline{R}(n) = R_3(\mathcal{B}, n) \) as in the proof of Theorem 1. Then we have

\[
R_3(n) = \sum_{1 \leq i \leq n/2} (1 - \eta(i)) (1 - \eta(n - i)) \\
= \sum_{1 \leq i \leq n/2} 1 - B(n - 1) - \eta(n/2) + \overline{R}(n)
\]

(here we have \( \eta(n/2) = 0 \) if \( n \) is odd). It follows that

\[
R_3(2k) = k - B(2k - 1) - \eta(k) + \overline{R}(2k)
\]

and

\[
R_3(2k + 1) = k - B(2k) + \overline{R}(2k + 1)
\]

then

\[
R_3(2k + 1) - R_3(2k) = -\left(B(2k) - B(2k - 1)\right) + \eta(k) + \left(\overline{R}(2k + 1) - \overline{R}(2k)\right)
\]

(18)

Clearly we have \( R_3(\mathcal{B}, 2k + 1) = R_2(\mathcal{B}, 2k + 1) \), and \( R_3(\mathcal{B}, 2k) - \eta(k) = R_2(\mathcal{B}, 2k) \) (if \( k \in \mathcal{B} \), then \( b = k, b' = k \) is a solution of \( b + b' = 2k \), \( b, b' \in \mathcal{B} \) \( b \leq b' \)) thus (18) can be rewritten as

\[
R_3(2k + 1) - R_3(2k) = -\eta(2k) + \left(R_2(\mathcal{B}, 2k + 1) - R_2(\mathcal{B}, 2k)\right)
\]

(19)

\[
\leq R_2(\mathcal{B}, 2k + 1) - R_2(\mathcal{B}, 2k).
\]

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It follows from (13) and (19) that

\[ 0 \leq -\eta(2k) + \left( R_2(\mathcal{B}, 2k + 1) - R_2(\mathcal{B}, 2k) \right) \leq R_2(\mathcal{B}, 2k + 1) - R_2(\mathcal{B}, 2k) \text{ for } k \geq n_0/2. \]  

(20)

Write \( \mathcal{B}_0 = \{ b : b \in \mathcal{B}, b + 1 \notin \mathcal{B}, 2 \mid b \} \), \( \mathcal{B}_1 = \{ b : b \in \mathcal{B}, b + 1 \notin \mathcal{B}, 2 \nmid b \} \). For a set \( \mathcal{S} \), define \( S(m, n) = \{ b : m \leq b \leq n, b \in \mathcal{S} \} \) and \( S(n) = S(1, n) \). By (12) we have at least one of \( \mathcal{B}_0 \) and \( \mathcal{B}_1 \) is an infinite set. Write

\[
M = \begin{cases} 
\max_{b \in \mathcal{B}_0} b & \text{if } |\mathcal{B}_0| < \infty \\
\max_{b \in \mathcal{B}_1} b & \text{if } |\mathcal{B}_1| < \infty \\
1 & \text{others.}
\end{cases}
\]

By Theorem G, there exists a constant \( C = C(\mathcal{A}) \) such that

\[ B(n) \leq C \log n \]

for infinitely many positive integers \( n \). By the bipartite method, there are infinitely many positive integers \( n \) with

\[ |B(n, 2n)| \leq 2C. \]

For such an integer \( n \), let \( b_u \) be the least \( b \in \mathcal{B} \) with \( b \geq 2n \). Then

\[ |B(\frac{1}{2} b_u, b_u)| \leq 2C + 1. \]

(21)

for large \( n \). Thus, there are infinitely many \( b_u \in \mathcal{B} \) with (21). Let \( b_u \) be such one with \( b_u > M + n_0 \) and \( b_{u+1} - b_u > 1 \), and let \( i = 0 \) or \( 1 \) with \( b_u \in \mathcal{B}_i \). Let

\[ v = v(u) = \min_{m \geq B(b_u - b_{u-1})} \{ b_m - b_{m-1} \} - 2 \]

and

\[ \mathcal{B}_i(v) = \{ \bar{b}_1 < \bar{b}_2 < \cdots < \bar{b}_x \}. \]

By the definition of \( M \) and (12), we have \( |\mathcal{B}_i(v)| \to \infty \) as \( u \to \infty \). So \( x > 2C + 1 \) for large \( u \). Since \( u = B(b_u) \geq B(b_u - b_{u-1}) \), we have

\[ \bar{b}_j \leq v < b_u - b_{u-1} \leq b_u . \]

So

\[ R_2(\mathcal{B}, b_u + \bar{b}_j) \geq 1 \quad \text{for } j = 1, 2, \ldots, x. \]

Noting that \( b_u, \bar{b}_j \in \mathcal{B}_i \), we have \( 2 \mid b_u + \bar{b}_j \). By \( b_u + \bar{b}_j \geq b_u > n_0 \) and (20), we have

\[ R_2(\mathcal{B}, b_u + \bar{b}_j + 1) \geq 1 \quad \text{for } j = 1, 2, \ldots, x. \]
Let
\begin{equation}
    b_u + b_j + 1 = b_s_j + b_t_j, \quad b_s_j < b_t_j, \quad j = 1, 2, \ldots, x. \tag{22}
\end{equation}

Then
\[ b_{t_j} > \frac{1}{2} (b_{s_j} + b_{t_j}) = \frac{1}{2} (b_u + b_j + 1) > \frac{1}{2} b_u \]
and
\[ b_{t_j} < b_u + b_j + 1 \leq b_u + v + 1 < b_u + b_{u+1} - b_u = b_{u+1}. \]

So
\[ b_{t_j} \in B\left(\frac{1}{2} b_u, b_u\right). \]

By (21), and \( x > 2C + 1 \), there exist \( 1 \leq p < q \leq x \) with \( t_p = t_q \). Hence, by (22), we have
\[ 0 < b_{s_q} - b_{s_p} = \overline{b}_q - \overline{b}_p \leq v. \]

So
\begin{equation}
    b_{s_{p+1}} - b_{s_p} \leq v. \tag{23}
\end{equation}

If \( b_{t_p} = b_u \), then \( b_{s_p} = \overline{b}_p + 1 \), a contradiction with \( \overline{b}_p \in B_t \). Thus, \( b_{t_p} < b_u \) and
\[ b_{s_p} = b_u + \overline{b}_p + 1 - b_{t_p} > b_u - b_{u-1}, \]
then \( s_p \geq B(b_u - b_{u-1}) \), a contradiction with (23) and the definition of \( v \). This completes the proof of Theorem 2.

\section*{References}

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