A NOTE ON THE LATTICE OF DENSITY PRESERVING MAPS

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We study here the poset $DP(X)$ of density preserving continuous maps defined on a Hausdorff space $X$ and show that it is a complete lattice for a compact Hausdorff space without isolated points. We further show that for countably compact $T_3$ spaces $X$ and $Y$ without isolated points, $DP(X)$ and $DP(Y)$ are order isomorphic if and only if $X$ and $Y$ are homeomorphic. Finally, Magill’s result on the remainder of a locally compact Hausdorff space is deduced from the relation of $DP(X)$ with posets $IP(X)$ of covering maps and $E_K(X)$ of compactifications respectively.

0. INTRODUCTION

Throughout the spaces considered (usually denoted by symbols $X, Y$) are Hausdorff and the maps are continuous. A map $f : X \to Y$ is called a density preserving map if $\text{Int} \ \text{Cl} f(A) \neq \phi$, whenever $\text{Int} A \neq \phi$, $A \subseteq X$ ([1]). Two density preserving maps $f$ and $g$ with domain $X$ and range $Rf$ and $Rg$ respectively are said to be equivalent ($f \approx g$) if there exists a homeomorphism $h : Rf \to Rg$ satisfying $h \circ f = g$. We identify equivalent density preserving maps on a fixed domain $X$, and denote by $DP(X)$ the set of all such equivalent classes of density preserving maps. The relation ‘$\leq$’ defined on $DP(X)$ by $g \leq f$ if there exists a continuous map $h : Rf \to Rg$ such that $h \circ f = g$ turns out to be a partial order relation. Recall that a perfect irreducible continuous surjection is called a covering map. In Section 1 we prove that if $X$ is a compact space without isolated points, then $DP(X)$ is a complete lattice. In Section 2, we determine the order structure of $DP(X)$ by proving that for countably compact $T_3$ spaces $X$ and $Y$ without isolated points, $DP(X)$ and $DP(Y)$ are order isomorphic if and only if $X$ and $Y$ are homeomorphic. Section 3 is devoted to the natural relation of $DP(X)$ with the poset $IP(X)$ of covering maps on $X$ ([3]) and the poset $E_K(X)$ of compactifications of a locally compact space $X$ ([2]). We show that if $U$ is an open dense set in a compact space $X$ then $DP(X, U) = IP(X, U)$, where $IP(X, U)$ (respectively $DP(X, U)$) is the poset of all covering (respectively density preserving-) maps $f$ on $X$ satisfying $|f^{-1}(f(x))| = 1$ for each $x$ in $U$. Using this result we deduce Magill’s result which states that for locally compact spaces $X$ and $Y$, $E_K(X)$ and $E_K(Y)$ are order isomorphic if and only if $\beta X - X$ and $\beta Y - Y$ are homeomorphic ([2]).

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1. Lattice \( DP(X) \)

We immediately have the following lemmas.

**Lemma 1.1.** \( DP(X) \) is a partially ordered set.

**Lemma 1.2.** Let \( f, g \in DP(X) \) be such that \( g \leq f \). Then the map \( h : Rf \rightarrow Rg \) satisfying \( h \circ f = g \) is a density preserving map.

**Proof:** Let \( A \subseteq Rf \) be such that \( \text{Int} A \neq \emptyset \). Then by setting \( f^{-1}(A) = A^* \), we get \( \emptyset \neq \text{Int} \text{Cl}(A^*) = \text{Int} \text{Cl}(h \circ f)(A^*) \subseteq \text{Int} \text{Cl}(A) \). Hence \( h \) is a density preserving map.

**Remark 1.3.** Fibres of a surjective density preserving map \( f : X \rightarrow Y \) are closed nowhere dense subsets of \( X \), where \( X \) is a space without isolated points.

**Definition 1.4:** For \( f \in DP(X) \), define \( \wp(f) = \{ f^{-1}(y) | y \in Rf \} \).

From here onwards we assume that members of \( DP(X) \) are quotient maps. If \( X \) is compact, this condition is automatically satisfied.

**Lemma 1.5.** Let \( f, g \in DP(X) \). Then \( f \leq g \) if and only if \( \wp(g) \subseteq \wp(f) \).

**Proof:** Let \( f \leq g \) then there exists \( h : Rg \rightarrow Rf \) satisfying \( h \circ g = f \). If \( g^{-1}(y) = A \in \wp(g) \) and if \( h(y) = x \), then \( A \subseteq (h \circ g)^{-1}(x) = f^{-1}(x) \). Conversely, suppose \( \wp(g) \subseteq \wp(f) \), then for \( z \in Rg \) take the unique \( y \in Rf \) for which \( g^{-1}(z) \subseteq f^{-1}(y) \) and define \( h : Rg \rightarrow Rf \) by \( h(z) = y \). Clearly \( h \) is continuous, \( h \circ g = f \) and hence \( f \leq g \).

**Note 1.6.** Two maps \( f \) and \( g \) are equivalent if and only if \( \wp(f) = \wp(g) \).

**Lemma 1.7.** Let \( X \) be a compact space without isolated points. Then \( DP(X) \) is a complete upper semi-lattice.

**Proof:** Let \( S \) be a non-empty subset of \( DP(X) \) and let \( Z = \prod \{ Rf | f \in S \} \). Consider the natural evaluation map \( g : X \rightarrow Z \) such that \( \pi_f(g(p)) = f(p) \), where \( \pi_f : Z \rightarrow Rf \) is the \( f \)-th projection map. Set \( T = g(X) \), \( \pi_f = \pi_f|T \) and define \( g' : X \rightarrow T \) by \( g'(p) = g(p) \), \( p \in X \). It is easy to verify that \( g' \) is the least upper bound of \( S \).

**Theorem 1.8.** Let \( X \) be a compact space without isolated points. Then \( DP(X) \) is a complete lattice.

**Proof:** Since a constant map onto its image is a density preserving map and any two such maps are equivalent, \( DP(X) \) has the minimum element. The required result now follows from Lemma 1.7 and the fact that a complete upper semilattice with minimum element is a complete lattice.
2. Order structure of $DP(X)$

The order structure of the poset $DP(X)$ is always determined by the topology on $X$, that is, if spaces $X$ and $Y$ are homeomorphic then $DP(X)$ and $DP(Y)$ are order isomorphic. We show here that the converse is true when $X$ and $Y$ are countably compact $T_3$ spaces without isolated points. The following terms and results are along the lines of [2, Lemmas 6, 9 and 10]. Throughout this section, our spaces are without isolated points.

**Definition 2.1:** A map $f \in DP(X)$ is said to be

(i) primary if $\varphi(f)$ has at most one non-singleton member.

(ii) dual if it is primary and $\varphi(f)$ contains exactly one doubleton.

**Notation.** If for some $f \in DP(X)$, $\varphi(f)$ contains $n$ non-singleton members, say $K_1, K_2, \ldots, K_n$, then $f$ is denoted by $(f, K_1, K_2, \ldots, K_n)$. In particular, if $K$ is a non-singleton closed nowhere dense set in $X$, then $(f, K)$ denotes the natural density preserving map defined on $X$ obtained by collapsing $K$ to a point.

**Lemma 2.2.**

I A map $f \in DP(X)$, $f \neq id_X$ is primary (respectively dual) if and only if there do not exist dual points $g, h \in DP(X)$ (respectively $g \in DP(X)$) such that $f \wedge g = f \wedge h \neq f$ and the only dual points greater than $g \wedge h$ are $g$ and $h$ (respectively $f < g < id_X$).

II For two closed nowhere dense subsets $K_1$ and $K_2$ of $X$,

$$(f, K_1) \wedge (g, K_2) = \begin{cases} (h, K_1, K_2), & \text{if } K_1 \cap K_2 = \emptyset \\ (h, K_1 \cup K_2), & \text{if } K_1 \cap K_2 \neq \emptyset. \end{cases}$$

III An order isomorphism $\varphi : DP(X) \to DP(Y)$ maps dual points to dual points.

**Definition 2.3:** A bijection $f : X \to Y$ is called a cln-bijection if $\{f(A) \mid A$ is a closed nowhere dense subset of $X\} = \{B \mid B$ is closed nowhere dense subset of $Y\}$.

**Lemma 2.4.** Let $\varphi : DP(X) \to DP(Y)$ be an order isomorphism. Then there exists a cln-bijection $F : X \to Y$ such that $f \in DP(X)$ implies $\varphi(\varphi(f)) = \{F(A) \mid A \in \varphi(f)\}$.

**Proof:** Take $p \in X$ and choose distinct points $q, r \in X - \{p\}$. By Lemma 2.2(III), $\varphi(f, \{p, q\})$, $\varphi(g, \{p, r\})$ are dual points of $DP(Y)$ say $(\overline{f}, \{a, b\})$ and $(\overline{g}, \{c, d\})$ respectively. Clearly $(\overline{f}, \{a, b\}) \wedge (\overline{g}, \{c, d\}) = \varphi(f \wedge g, \{p, q, r\})$. If $\{a, b\} \cap \{c, d\} = \emptyset$, then $(\overline{f}, \{a, b\}) \wedge (\overline{g}, \{c, d\}) = (\overline{f} \wedge \overline{g}, \{a, b\}, \{c, d\}); (f, \{p, q\}), (g, \{p, r\}), (h\{q, r\})$ are three dual points greater than $(f \wedge g, \{p, q, r\})$ and $(\overline{f}, \{a, b\}), (\overline{g}, \{c, d\})$.
are two dual points greater than \((f, H)\) which is not possible. Therefore \(\{a, b\} \cap \{c, d\} \neq \emptyset\), in fact it is a singleton, say \(\{a\}\). Define \(F : X \rightarrow Y\) by \(F(p) = a\). Note that the choice of \(a\) does not depend on the choice of \(r\) and \(q\). In general, if \(f \in DP(X)\) is of the form \((f, H)\) and if \(\varphi(f, H) = \bar{f}\), then it is easy to verify that \(\bar{f} = (\bar{f}, K)\) for some closed nowhere dense subset \(K\) of \(Y\). Further, if \(p, q \in H\), \(p \neq q\) then \((g, \{p, q\}) \geq (f, H)\) which implies \((\bar{g}, \{a, b\}) \geq (\bar{f}, K)\) therefore \(F(\{p, q\}) = \{a, b\} \subseteq K\) and hence \(F(H) \subseteq K\). Similarly we can use \(\varphi^{-1}\) to define \(\bar{F} : Y \rightarrow X\) and obtain \(\bar{F}(K) \subseteq H\). Observe that \(\bar{F} \circ F\) is identity on \(X\). In fact, if \(p \in X\) and \(q \in X - \{p\}\), then \(\varphi(f, \{p, q\})\) is dual point say \((\bar{f}, \{a, b\})\) and \(F(p) \in \{a, b\}\). Assume \(F(p) = a\). Then \(\bar{F}(a) = q\). Choose \(r \in X - \{q, p\}\) then there exists \(c \in Y\) such that \(\varphi(g, \{p, r\})\) is a dual point say \((\bar{g}, \{a, c\})\). Since \(\bar{F}(a) \in \{p, r\}\) and \(\bar{F}(a) \neq p\), therefore \(\bar{F}(a) = r\), a contradiction. Similarly, \(F \circ \bar{F}\) is identity on \(Y\). We have also shown in the process that if \(\varphi(f, H) = (\bar{f}, K)\), then \(F(H) = K\).

Recall that a subset \(A\) of countably compact \(T_3\) space \(X\) without isolated points is closed if and only if whenever \(B \subseteq A\) and \(\text{Cl}_X B\) is nowhere dense in \(X\) then \(\text{Cl}_X B \subseteq A\). Using this fact, Lemma 2.4 and the technique of [3, Theorem 1.1], we have the following.

**Theorem 2.5.** Let \(X\) and \(Y\) be countably compact \(T_3\) spaces without isolated points. Then \(DP(X)\) and \(DP(Y)\) are order isomorphic if and only if \(X\) and \(Y\) are homeomorphic.

**Note 2.6.** The map \(f : Q \cup \{p\} \rightarrow Q \cup \{q\}\) in [3, example 3.9] defined by \(f(x) = x\) if \(x \in Q\) and \(f(p) = q\), where \(p\) and \(q\) are remote points of \(Q\) such that Stone’s extension of no self-homeomorphism of \(Q\) maps \(p\) to \(q\), is a cln-bijection between non countably compact spaces which is not a homeomorphism.

### 3. \(DP(X)\) and \(IP(X)\)

**Definition 3.1:** For a subset \(A\) in \(X\) we define

\[
DP(X, A) = \left\{ f \in DP(X) \mid \left| f^{-1}(f(x)) \right| = 1, \text{ for all } x \in A \right\}.
\]

**Note 3.2.**

(i) \(DP(X, A)\) is a poset with respect to the order defined on \(DP(X)\).

(ii) If \(g \in DP(X, A), f \in DP(X)\) and \(g \leq f\), then \(f \in DP(X, A)\).

**Theorem 3.3.** Let \(A\) be a subset of a compact space \(X\) containing all isolated points of \(X\). The \(DP(X, A)\) is a complete upper semilattice.

**Proof:** Follows from Lemma 1.7 and Note 3.2(ii).
Theorem 3.4. Let $A_i$ be any subset of $X_i$ containing all isolated points of $X_i$, $i = 1, 2$ and \(\varphi : DP(X_1, A_1) \to DP(X_2, A_2)\) be an order isomorphism. Then there is a cln-bijection \(F : X_1 - A_1 \to X_2 - A_2\).

Proof: Follows along the lines of Lemma 2.4.

Theorem 3.5. Let $A$ be a dense subspace of a space $X$. Then every $f$ in $DP(X, A)$ is irreducible.

Proof: Let \(f \in DP(X, A)\). $F$ be a proper closed subset of $X$ and \(f(F) = Rf\). Then for every \(y \in (X - F) \cap A\), \(\left|f^{-1}(f(y))\right| \neq 1\) which contradicts the choice of $f$.

Corollary 3.6. If $X$ is compact and $A$ is dense in $X$ then $DP(X, A) = IP(X, A)$. In particular, if $X$ is locally compact the $DP(\alpha X, X) = IP(\alpha X, X)$, where $\alpha X$ is a compactification of $X$.

Proof: Set $D_C(X, A) = \{f \in DP(X, A) \mid f \text{ is closed}\}$. Observe that $D_C(X, A) \subseteq IP(X)$ and $D_C(X, A) = DP(X, A)$.

Note 3.7. In general, if $A$ is not dense then $D_C(X, A) \subseteq IP(X)$ need not be true. For example take $X = [0, 1]$, $A = [0, 1/2)$ and define $f : X \to X$ by

\[
    f(x) = \begin{cases} 
        2x, & 0 \leq x \leq \frac{1}{2} \\
        3 - x, & \frac{1}{2} \leq x \leq 1
    \end{cases}
\]

Clearly $f \in D_C(X, A) - IP(X)$.

We recall the following result [3, Lemma 3.11].

Lemma 3.8. Let $X$ be a locally compact space. The function $\psi : IP(\beta X, X) \to E_K(X)$ defined by $\psi(f) = \beta X \mid_\varphi(f)$ is an order isomorphism, where $\beta X \mid_\varphi(f)$ is the natural compactification of $X$ obtained by collapsing each fibre in $\varphi(f)$ to a point.

We now deduce following result due to Magill [2, Theorem 12].

Theorem 3.9. Let $X$ and $Y$ be locally compact spaces. Then $E_K(X)$ and $E_K(Y)$ are order isomorphic if and only if $\beta X - X$ and $\beta Y - Y$ are homeomorphic.

Proof: If $E_K(X)$ and $E_K(Y)$ are ordered isomorphic, then by Corollary 3.6 and Lemma 3.8, $DP(\beta X, X)$ and $DP(\beta Y, Y)$ are order isomorphic and hence Theorem 3.4 gives a cln-bijection $F : \beta X - X \to \beta Y - Y$. Since all closed subsets in $\beta X - X$ are nowhere dense, $F$ is a closed map. Similarly $F^{-1}$ is also a closed map.

References


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