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INEQUALITIES FOR THE BETA FUNCTION OF *n* VARIABLES

HORST ALZER[1]

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Abstract

We present various inequalities for Euler's beta function of n variables. One of our theorems states that the inequalities

$$a_n \leq \frac{1}{\prod_{i=1}^{n} x_i} - B(x_1, \dots, x_n) \leq b_n \qquad (*)$$

hold for all $x_i \geq 1$ ($i = 1, \dots, n$; $n \geq 3$) with the best possible constants $a_n = 0$ and $b_n = 1 - 1/(n-1)!$. This extends a recently published result of Dragomir *et al.*, who investigated ($*$) for the special case $n = 2$.

1. Introduction

The classical beta function, which is also known as Euler's integral of the first kind, is defined for positive real numbers x and y by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt. \qquad (1.1)$$

The beta function plays a central role in the theory of special functions and also has applications in other fields, such as mathematical physics and probability theory; see [4, 5, 8]. An extension of (1.1) to n variables is given by

$$B(x_1, \dots, x_n) = \int_{\Delta_{n-1}} \left(\prod_{i=1}^{n-1} t_i^{x_i - 1}\right)\left(1 - \sum_{i=1}^{n-1} t_i\right)^{x_n - 1} dt_1 \cdots dt_{n-1}$$

($x_i > 0$; $i = 1, \dots, n$; $n \geq 2$), where

$$\Delta_{n-1} = \left\{(t_1, \dots, t_{n-1}) \in \mathbf{R}^{n-1} | t_1 \geq 0, \dots, t_{n-1} \geq 0, t_1 + \cdots + t_{n-1} \leq 1\right\}$$

[1]Morsbacher Str. 10, 51545 Waldbröl, Germany; e-mail: alzerhorst@freenet.de.

denotes the standard simplex in $\mathbf{R}^{n-1}$. There exists a close connection between $B(x_1, \dots, x_n)$ and the gamma function,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad (x > 0),$$

as the elegant identity

$$B(x_1, \dots, x_n) = \frac{\Gamma(x_1) \cdots \Gamma(x_n)}{\Gamma(x_1 + \cdots + x_n)}$$

reveals. A collection of the most important properties of the beta function of two and more variables is given, for instance, in [4, 8].

Various inequalities for $B(x, y)$ and $B_p(x, y) = \int_0^p t^{x-1}(1-t)^{y-1}\, dt$ appear in the literature (see [12, 13, 15, 16, 19]), whereas inequalities for the beta function of three or more variables are difficult to find. The following interesting inequality for $B(x, y)$ was published in 2000 by Dragomir *et al.* [9]:

$$0 \le 1/(xy) - B(x, y) \le 1/4 \quad (x, y \ge 1). \tag{1.2}$$

The lower bound 0 is sharp, but the upper bound 1/4 can be improved. In [3] it is shown that the second inequality of (1.2) is valid with the best possible constant $0.08731\dots$. It is natural to look for an extension of (1.2) to more than two variables. In this paper we determine the best possible constants a_n and b_n such that the double-inequality ($*$) holds for all $x_i \ge 1$ ($i = 1, \dots, n; n \ge 3$). Furthermore, we establish several new inequalities for $B(x_1, \dots, x_n)$, which are valid for all $n \ge 2$. In Section 3 we provide sharp constants $\alpha_n(c)$ and $\beta_n(c)$ in

$$\alpha_n(c) \frac{\prod_{i=1}^n x_i^{-1/2+x_i}}{\left(\sum_{i=1}^n x_i\right)^{-1/2+\sum_{i=1}^n x_i}} \le B(x_1, \dots, x_n) \le \beta_n(c) \frac{\prod_{i=1}^n x_i^{-1/2+x_i}}{\left(\sum_{i=1}^n x_i\right)^{-1/2+\sum_{i=1}^n x_i}},$$

where $x_i \ge c > 0$ ($i = 1, \dots, n$). Moreover, we determine the best possible upper and lower bounds for the ratio $B(\mu x_1, \mu x_2, \dots, \mu x_n)/B(\nu x_1, \nu x_2, \dots, \nu x_n)$, depending only on μ, ν and n, and we establish that the inequalities

$$B\big((x_1 + x_2)/2, \dots, (x_n + y_n)/2\big) \le \sqrt{B(x_1, \dots, x_n) B(y_1, \dots, y_n)}$$

and

$$B(x_1 + y_1, \dots, x_n + y_n) < \frac{1}{2^n}\big(B(x_1, \dots, x_n) + B(y_1, \dots, y_n)\big)$$

are valid for all $x_i > 0$ ($i = 1, \dots, n$). In order to prove our results we need some lemmas, which we present in the next section.

2. Lemmas

First, we collect a few basic properties of the gamma function and its logarithmic derivative $\psi = \Gamma'/\Gamma$, which is known as the psi or digamma function.

LEMMA 2.1. *Let $a > 0$, $b \geq 0$ and $x > 0$ be real numbers and let $n \geq 1$ be an integer. Then we have*

$$\Gamma(ax+b) \sim \sqrt{2\pi}\, e^{-ax}(ax)^{ax+b-1/2} \quad (x \to \infty), \tag{2.1}$$

$$\log \Gamma(x) \sim (x-1/2)\log x - x + \frac{1}{2}\log(2\pi) + \frac{1}{12x} - \cdots \quad (x \to \infty), \tag{2.2}$$

$$\Gamma(2x) = \frac{1}{2\sqrt{\pi}} 4^x \Gamma(x)\Gamma(x+1/2), \tag{2.3}$$

$$\lim_{x\to\infty} x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1, \tag{2.4}$$

$$\psi(x+1) = \psi(x) + 1/x, \tag{2.5}$$

$$\psi(x) \sim \log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \cdots \quad (x \to \infty), \tag{2.6}$$

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty e^{-xt}\frac{t^n}{1-e^{-t}}\,dt = (-1)^{n+1} n! \sum_{k=0}^{\infty}\frac{1}{(x+k)^{n+1}}, \tag{2.7}$$

$$\frac{1}{x} + \frac{1}{2x^2} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}. \tag{2.8}$$

The formulas (2.1)–(2.7) can be found in [1], while (2.8) and corresponding rational bounds for $\psi^{(n)}$ with $n \geq 2$ are given in [2, 10]. The following two lemmas present inequalities for the psi function.

LEMMA 2.2. *Let $t \geq 3$ be a real number and let $a = 1 - 1/\Gamma(t)$. Then we have for all real numbers $x \geq 1$:*

$$0 < ax^{t-1} + (ax^t - 1)\psi(tx). \tag{2.9}$$

PROOF. We denote the expression on the right-hand side of (2.9) by $f(x)$. Differentiation gives

$$xf'(x) = g(x) + a(t-1)x^{t-1} + atx^{t+1}\psi'(tx), \tag{2.10}$$

where $g(x) = atx^t\psi(tx) - tx\psi'(tx)$. Since ψ is positive on $(1.461\ldots, \infty)$, $x \geq 1$ and $a \geq 1/2$ imply $g(x)/(tx) \geq \psi(tx)/2 - \psi'(tx)$. Since ψ and $-\psi'$ are strictly increasing on $(0, \infty)$, we obtain

$$\frac{g(x)}{tx} \geq \frac{1}{2}\psi(3) - \psi'(3) = 0.066\ldots. \tag{2.11}$$

From (2.10) and (2.11) we conclude that $f'(x) > 0$ for $x \geq 1$. Hence we have

$$f(x) \geq f(1) = a + (a-1)\psi(t) = \frac{h(t)}{\Gamma(t)}, \tag{2.12}$$

where $h(t) = \Gamma(t) - \psi(t) - 1$. Differentiation yields $h'(t) = \Gamma'(t) - \psi'(t)$ and $h''(t) = \Gamma''(t) - \psi''(t)$. Since Γ'' and $-\psi''$ are positive on $(0, \infty)$, we obtain for $t \geq 3$: $h'(t) \geq h'(3) = 1.450\ldots$ and $h(t) \geq h(3) = 0.077\ldots$, so that (2.12) implies that f is positive on $[1, \infty)$.

LEMMA 2.3. *Let $n \geq 3$ be an integer and let $a = 1 - 1/\Gamma(n)$. Then we have for all real numbers $x_i \geq 1$ $(i = 1, \ldots, n)$:*

$$0 < \psi\left(\sum_{i=1}^{n} x_i\right)\left[a\prod_{i=1}^{n} x_i - 1\right] + a\left(\max_{1\leq i\leq n} x_i\right)^{-1}\prod_{i=1}^{n} x_i.$$

PROOF. We may assume that $x_1 \geq \cdots \geq x_n \geq 1$. Let

$$f(x_1, \ldots, x_n) = \psi\left(\sum_{i=1}^{n} x_i\right)\left[a\prod_{i=1}^{n} x_i - 1\right] + a\prod_{i=2}^{n} x_i$$

and $f_q(x) = f(x, \ldots, x, x_{q+1}, \ldots, x_n)$, where $x > 0$ and $q \in \{1, \ldots, n-1\}$. We prove that f_q is increasing on $[x_{q+1}, \infty)$. Let $x \geq x_{q+1}$ and $y = qx + \sum_{i=q+1}^{n} x_i \geq n$. Differentiation gives

$$\frac{1}{q} f_q'(x) = \psi'(y)\left[ax^q\prod_{i=q+1}^{n} x_i - 1\right] + \psi(y)ax^{q-1}\prod_{i=q+1}^{n} x_i + a(1-1/q)x^{q-2}\prod_{i=q+1}^{n} x_i.$$

Since $x^q\prod_{i=q+1}^{n} x_i \geq 1$, $a > 0$, $\psi(y) > 0$, and $\psi'(y) > 0$, we obtain

$$\frac{1}{q} f_q'(x) \geq (a-1)\psi'(y) + a\psi(y) = g(y), \quad \text{say}.$$

The functions $(a-1)\psi'$ and $a\psi$ are strictly increasing on $(0, \infty)$, so that we get

$$\Gamma(n)g(y) \geq \Gamma(n)g(n) = \psi(n)[\Gamma(n) - 1] - \psi'(n) = h(n), \quad \text{say}.$$

Since $h'(n) = \psi'(n)[\Gamma(n) - 1] + \psi(n)\Gamma'(n) - \psi''(n) > 0$ for $n \geq 3$, we obtain

$$h(n) \geq h(3) = 0.527\ldots.$$

This implies that $f_q'(x) > 0$ for $x \geq x_{q+1}$. Thus we get

$$\begin{aligned} f(x_1, \ldots, x_n) = f_1(x_1) \geq f_1(x_2) = f_2(x_2) \geq f_2(x_3) \geq \cdots \geq f_{n-1}(x_n) \\ = ax_n^{n-1} + (ax_n^n - 1)\psi(nx_n). \end{aligned}$$

Applying Lemma 2.2 we conclude that $f(x_1, \ldots, x_n) > 0$.

Further, we need the following monotonicity theorem.

LEMMA 2.4. *Let $a > 1$ be a real number. The function*

$$\phi_a(x) = a(x - 1/2)\log x - (ax - 1/2)\log(ax) - a\log\Gamma(x) + \log\Gamma(ax) \quad (2.13)$$

is strictly increasing on $(0, \infty)$ *with* $\lim_{x\to\infty}\phi_a(x) = -\frac{1}{2}(a-1)\log(2\pi)$.

PROOF. Let $x > 0$. Differentiation gives

$$x\phi_a'(x) = -\frac{a-1}{2} - ax\log a - ax\psi(x) + ax\psi(ax) = p_a(x), \quad \text{say.} \quad (2.14)$$

Further, we get

$$\frac{1}{a}p_a'(x) = -\log a - \psi(x) + \psi(ax) - x\psi'(x) + ax\psi'(ax) \quad (2.15)$$

and

$$\frac{x}{a}p_a''(x) = q(ax) - q(x), \quad (2.16)$$

where $q(x) = 2x\psi'(x) + x^2\psi''(x)$. Next, we prove that q is strictly increasing on $(0, \infty)$. We obtain

$$\frac{1}{x^2}q'(x) = \frac{2}{x^2}\psi'(x) + \frac{4}{x}\psi''(x) + \psi'''(x).$$

Using the integral formulas (2.7) and

$$\frac{1}{x^n} = \frac{1}{(n-1)!}\int_0^\infty e^{-xt}t^{n-1}\,dt \quad (x > 0; n = 1, 2, \dots),$$

and the convolution theorem for Laplace transforms, we get

$$\frac{1}{x^2}q'(x) = \int_0^\infty e^{-xt}\Lambda(t)\,dt, \quad (2.17)$$

where

$$\Lambda(t) = 2t\int_0^t \frac{s}{1-e^{-s}}\,ds - 6\int_0^t \frac{s^2}{1-e^{-s}}\,ds + \frac{t^3}{1-e^{-t}}.$$

Let $t > 0$. Then we obtain

$$\Lambda'(t) = 2\int_0^t \frac{s}{1-e^{-s}}\,ds - \left(\frac{t}{1-e^{-t}}\right)^2[1 + e^{-t}(t-1)]$$

and

$$\frac{(1-e^{-t})^3}{t^2}e^{2t}\Lambda''(t) = 2+t+(t-2)e^t = \sum_{k=3}^{\infty}\frac{k-2}{k!}t^k > 0.$$

Since $\Lambda(0) = \Lambda'(0) = 0$, we get $\Lambda(t) > 0$ for $t > 0$. From (2.17) we conclude that q is strictly increasing on $(0, \infty)$, so that (2.16) implies $p_a''(x) > 0$ for $x > 0$. Using the asymptotic expansion (2.6) and the limit relation $\lim_{x\to\infty} x\psi'(x) = 1$, we conclude from (2.14) and (2.15) that $\lim_{x\to\infty} p_a(x) = \lim_{x\to\infty} p_a'(x) = 0$. Thus p_a is positive on $(0, \infty)$. From (2.14) we obtain that ϕ_a is strictly increasing on $(0, \infty)$.

The asymptotic formula (2.2) implies $\lim_{x\to\infty}\phi_a(x) = -\frac{1}{2}(a-1)\log(2\pi)$.

3. Main results

We are now in a position to prove the inequalities for the beta function that we announced in Section 1. Our first theorem provides a generalisation of the double-inequality (1.2).

THEOREM 3.1. *Let $n \geq 3$ be an integer. Then we have for all real numbers $x_i \geq 1$ $(i = 1, \dots, n)$:*

$$0 < \frac{1}{\prod_{i=1}^n x_i} - B(x_1, \dots, x_n) \leq 1 - \frac{1}{(n-1)!}. \tag{3.1}$$

Both bounds are best possible.

PROOF. The first inequality of (3.1) is equivalent to

$$0 < \log\Gamma(x_1 + \cdots + x_n) - \sum_{i=1}^n \log\Gamma(x_i + 1). \tag{3.2}$$

To prove (3.2) we may assume that $x_1 \geq \cdots \geq x_n \geq 1$. We denote the right-hand side of (3.2) by $f(x_1, \dots, x_n)$. Further, let $q \in \{1, \dots, n-1\}$, $x \geq x_{q+1}$, and

$$\begin{aligned} f_q(x) &= f(x, \dots, x, x_{q+1}, \dots, x_n) \\ &= \log\Gamma\left(qx + \sum_{i=q+1}^n x_i\right) - q\log\Gamma(x+1) - \sum_{i=q+1}^n \log\Gamma(x_i+1). \end{aligned}$$

Since ψ is strictly increasing on $(0, \infty)$, we get

$$\frac{1}{q}f_q'(x) = \psi\left(qx + \sum_{i=q+1}^n x_i\right) - \psi(x+1) > 0,$$

so that f_q is strictly increasing on $[x_{q+1}, \infty)$. This implies

$$\begin{aligned} f(x_1, \dots, x_n) &= f_1(x_1) \geq f_1(x_2) = f_2(x_2) \geq f_2(x_3) \geq \cdots \geq f_{n-1}(x_n) \\ &= \log \Gamma(nx_n) - n \log \Gamma(x_n + 1). \end{aligned} \tag{3.3}$$

Let $g(x) = \log \Gamma(nx) - n \log \Gamma(x+1)$. Then we get for $x \geq 1$:

$$g'(x)/n = \psi(nx) - \psi(x+1) > 0$$

and

$$g(x) \geq g(1) = \log \Gamma(n) \geq \log \Gamma(3) = \log 2. \tag{3.4}$$

From (3.3) and (3.4) we conclude that (3.2) is valid.

Using the asymptotic formula (2.1) we obtain

$$\lim_{x \to \infty} \big(1/x^n - B(x, \dots, x)\big) = \lim_{x \to \infty} \big(1/x^n - \big(\Gamma(x)\big)^n / \Gamma(nx)\big) = 0,$$

which implies that in (3.1) the lower bound 0 cannot be replaced by a larger constant.

Let $a = 1 - 1/(n-1)!$. To prove the right-hand side of (3.1) we have to show that

$$0 \leq \Gamma\left(\sum_{i=1}^n x_i\right)\left[a \prod_{i=1}^n x_i - 1\right] + \prod_{i=1}^n \Gamma(x_i + 1) = u(x_1, \dots, x_n), \quad \text{say}.$$

Let $q \in \{1, \dots, n-1\}$, $x_1 \geq \cdots \geq x_n \geq 1$, and

$$\begin{aligned} u_q(x) &= u(x, \dots, x, x_{q+1}, \dots, x_n) \\ &= \Gamma\left(qx + \sum_{i=q+1}^n x_i\right)\left[ax^q \prod_{i=q+1}^n x_i - 1\right] + \big(\Gamma(x+1)\big)^q \prod_{i=q+1}^n \Gamma(x_i + 1). \end{aligned}$$

We set $y = qx + \sum_{i=q+1}^n x_i$ and apply Lemma 2.3. Then we get for $x \geq x_{q+1}$:

$$\begin{aligned} \big(q\Gamma(y)\big)^{-1} u_q'(x) &= \psi(y)\left[ax^q \prod_{i=q+1}^n x_i - 1\right] + ax^{q-1} \prod_{i=q+1}^n x_i \\ &\quad + \big(\Gamma(x+1)\big)^q \psi(x+1) \big(\Gamma(y)\big)^{-1} \prod_{i=q+1}^n \Gamma(x_i + 1) > 0. \end{aligned}$$

Hence u_q is strictly increasing on $[x_{q+1}, \infty)$. This implies

$$\begin{aligned} u(x_1, \dots, x_n) &= u_1(x_1) \geq u_1(x_2) = u_2(x_2) \geq \cdots \geq u_{n-1}(x_n) \\ &= (ax_n^n - 1)\Gamma(nx_n) + \big(\Gamma(x_n + 1)\big)^n. \end{aligned} \tag{3.5}$$

Let $v(x) = (ax^n - 1)\Gamma(nx) + \big(\Gamma(x+1)\big)^n$. Then we have

$$\frac{v'(x)}{n\Gamma(nx)} = ax^{n-1} + (ax^n - 1)\psi(nx) + \frac{\big(\Gamma(x+1)\big)^n \psi(x+1)}{\Gamma(nx)}.$$

From Lemma 2.2 we conclude that v is strictly increasing on $[1, \infty)$. Thus

$$v(x) \geq v(1) = 0 \quad \text{for} \quad x \geq 1,$$

so that (3.5) yields $u(x_1, \ldots, x_n) \geq 0$.

If $x_1 = \cdots = x_n = 1$, then the second inequality of (3.1) holds with equality. This implies that the upper bound $1 - 1/(n-1)!$ is sharp.

REMARK. The inequalities (3.1) are not valid for all positive real numbers x_i ($i = 1, \ldots, n$). More precisely: there do not exist constants $c_1(n)$ and $c_2(n)$ such that

$$c_1(n) \leq \frac{1}{\prod_{i=1}^n x_i} - B(x_1, \ldots, x_n) \leq c_2(n) \tag{3.6}$$

holds for all $x_i > 0$ ($i = 1, \ldots, n; n \geq 2$). Indeed, if we set $x_1 = \cdots = x_{n-1} = x > 0$ and $x_n = y > 1$, then the left-hand side of (3.6) yields

$$x^{n-1} y c_1(n) \leq 1 - \frac{(\Gamma(x+1))^{n-1}\Gamma(y+1)}{\Gamma((n-1)x + y)}.$$

We let x tend to 0 and obtain the incorrect inequality $0 \leq 1 - \Gamma(y+1)/\Gamma(y) = 1 - y$. And, if we set $x_1 = \cdots = x_n = x > 0$, then the right-hand side of (3.6) gives

$$\frac{1}{x^n} - \frac{(\Gamma(x))^n}{\Gamma(nx)} = \frac{\Gamma(nx+1) - nx(\Gamma(x+1))^n}{x^n \Gamma(nx+1)} \leq c_2(n).$$

This is false, since the term on the left-hand side tends to ∞, if we let x tend to 0.

The next theorem provides sharp upper and lower bounds for $B(x_1, \ldots, x_n)$, which are valid in $[c, \infty)^n$, where $c > 0$ is a fixed real number.

THEOREM 3.2. *Let $c > 0$ be a real number and let $n \geq 2$ be an integer. Then we have for all real numbers $x_i \geq c$ ($i = 1, \ldots, n$):*

$$\alpha_n(c) \frac{\prod_{i=1}^n x_i^{-1/2 + x_i}}{\left(\sum_{i=1}^n x_i\right)^{-1/2 + \sum_{i=1}^n x_i}} < B(x_1, \ldots, x_n) \leq \beta_n(c) \frac{\prod_{i=1}^n x_i^{-1/2 + x_i}}{\left(\sum_{i=1}^n x_i\right)^{-1/2 + \sum_{i=1}^n x_i}}, \tag{3.7}$$

with the best possible constants

$$\alpha_n(c) = (2\pi)^{(n-1)/2} \quad \textit{and} \quad \beta_n(c) = n^{nc - 1/2} c^{(n-1)/2} \frac{(\Gamma(c))^n}{\Gamma(nc)}. \tag{3.8}$$

PROOF. Let $x > 0$ and $x_i > 0$ $(i = 1, \dots, n)$ be real numbers and let $q \in \{1, \dots, n-1\}$. We define

$$f(x_1, \dots, x_n) = \sum_{i=1}^{n}(x_i - 1/2)\log x_i - \left(\sum_{i=1}^{n} x_i - \frac{1}{2}\right)\log\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n}\log\Gamma(x_i) + \log\Gamma\left(\sum_{i=1}^{n} x_i\right)$$

and

$$\begin{aligned} f_q(x) &= f(x, \dots, x, x_{q+1}, \dots, x_n) \\ &= q\left(x - \frac{1}{2}\right)\log x + \sum_{i=q+1}^{n}(x_i - 1/2)\log x_i \\ &\quad - \left(qx + \sum_{i=q+1}^{n} x_i - \frac{1}{2}\right)\log\left(qx + \sum_{i=q+1}^{n} x_i\right) - q\log\Gamma(x) \\ &\quad - \sum_{i=q+1}^{n}\log\Gamma(x_i) + \log\Gamma\left(qx + \sum_{i=q+1}^{n} x_i\right). \end{aligned}$$

Then we get $f_q'(x)/q = g(x) - g(y)$, where $g(z) = \log z - 1/(2z) - \psi(z)$ and $y = qx + \sum_{i=q+1}^{n} x_i$. The left-hand side of (2.8) implies $g'(z) = 1/z + 1/(2z^2) - \psi'(z) < 0$ for $z > 0$. Hence we conclude from $y > x$ that $g(y) < g(x)$. This implies that f_q is strictly increasing on $(0, \infty)$.

To prove the right-hand inequality of (3.7) with $\beta_n(c)$ as defined in (3.8), we assume that $x_1 \geq \cdots \geq x_n \geq c$. Then we obtain

$$\begin{aligned} f(x_1, \dots, x_n) = f_1(x_1) \geq f_1(x_2) = f_2(x_2) \geq f_2(x_3) \geq \cdots \geq f_{n-1}(x_n) \\ = \phi_n(x_n), \end{aligned} \tag{3.9}$$

where ϕ_n is defined in (2.13). From Lemma 2.4 we get

$$\phi_n(x_n) \geq \phi_n(c) = -\log\beta_n(c), \tag{3.10}$$

so that (3.9) and (3.10) lead to

$$f(x_1, \dots, x_n) \geq -\log\beta_n(c), \tag{3.11}$$

which is equivalent to the second inequality of (3.7). Moreover, since f_q and ϕ_n are strictly monotonic, we conclude that the sign of equality holds in (3.11) if and only if $x_1 = \cdots = x_n = c$.

To prove the left-hand side of (3.7) with $\alpha_n(c) = (2\pi)^{(n-1)/2}$ we suppose that $c \leq x_1 \leq \cdots \leq x_n$. The monotonicity of f_q and Lemma 2.4 lead to

$$f(x_1, \dots, x_n) = f_1(x_1) \leq f_1(x_2) = f_2(x_2) \leq f_2(x_3) \leq \cdots \leq f_{n-1}(x_n)$$

$$= \phi_n(x_n) < (-1/2)(n-1)\log(2\pi) = -\log\alpha_n(c),$$

which leads to the first inequality of (3.7) with $\alpha_n(c) = (2\pi)^{(n-1)/2}$.

Conversely, we assume that the left-hand inequality of (3.7) is valid for all $x_i \geq c$ $(i = 1, \dots, n)$. Then we set $x_1 = \cdots = x_n = x > 0$ and obtain $\alpha_n(c) < e^{-\phi_n(x)}$. Applying Lemma 2.4 we get $\alpha_n(c) \leq \lim_{x\to\infty} e^{-\phi_n(x)} = (2\pi)^{(n-1)/2}$. Thus in (3.7) the factor $\alpha_n(c) = (2\pi)^{(n-1)/2}$ cannot be replaced by a larger constant.

If a function f satisfies the inequality $f(\delta x_1, \dots, \delta x_n) \leq \delta f(x_1, \dots, x_n)$ for all $x_i > 0\,(i = 1, \dots, n)$ and $\delta \in (0, 1)$, then f is said to be starshaped on $\mathbf{R}_+^n$. Interesting properties of these functions can be found in [6, 7]. As an immediate consequence of the following theorem we obtain that the beta function is not starshaped on $\mathbf{R}_+^n$.

THEOREM 3.3. *Let μ and ν be real numbers with $\mu > \nu > 0$ and let $n \geq 2$ be an integer. Then we have for all real numbers $x_i > 0$ $(i = 1, \dots, n)$:*

$$0 < \frac{B(\mu x_1, \mu x_2, \dots, \mu x_n)}{B(\nu x_1, \nu x_2, \dots, \nu x_n)} < \left(\frac{\nu}{\mu}\right)^{n-1}. \tag{3.12}$$

Both bounds are best possible.

PROOF. To establish the second inequality of (3.12) it suffices to show that the function $f(t) = t^{n-1}B(tx_1, \dots, tx_n)$ is strictly decreasing on $(0, \infty)$. Let $t > 0$. Differentiation yields

$$\frac{t}{f(t)} f'(t) = n - 1 + \sum_{i=1}^{n} tx_i\psi(tx_i) - \psi\left(\sum_{i=1}^{n} tx_i\right)\sum_{i=1}^{n} tx_i. \tag{3.13}$$

We set $y_i = tx_i > 0$ $(i = 1, \dots, n)$ and define

$$g(y_1, \dots, y_n) = \psi\left(\sum_{i=1}^{n} y_i\right)\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} y_i\psi(y_i).$$

In order to prove

$$g(y_1, \dots, y_n) > n - 1 \tag{3.14}$$

we assume that $y_1 \geq \cdots \geq y_n > 0$. Let $q \in \{1, \dots, n-1\}$, $y > 0$,, and

$$\begin{aligned} g_q(y) &= g(y, \dots, y, y_{q+1}, \dots, y_n) \\ &= \psi\left(qy + \sum_{i=q+1}^{n} y_i\right)\left(qy + \sum_{i=q+1}^{n} y_i\right) - qy\psi(y) - \sum_{i=q+1}^{n} y_i\psi(y_i). \end{aligned}$$

Then we get

$$g_q'(y)/q = h(z) - h(y), \tag{3.15}$$

where

$$h(x) = \psi(x) + x\psi'(x) \quad \text{and} \quad z = qy + \sum_{i=q+1}^{n} y_i. \tag{3.16}$$

Using the series representation (2.7) we obtain

$$h'(x) = 2\psi'(x) + x\psi''(x) = 2\sum_{k=1}^{\infty} \frac{k}{(x+k)^3} > 0.$$

Since $z > y$, we get $h(z) > h(y)$, so that (3.15) implies that g_q is strictly increasing on $(0, \infty)$. Hence we have

$$\begin{aligned} g(y_1, \dots, y_n) &= g_1(y_1) \geq g_1(y_2) = g_2(y_2) \geq \cdots \geq g_{n-1}(y_n) \\ &= ny_n[\psi(ny_n) - \psi(y_n)]. \end{aligned} \tag{3.17}$$

Let

$$\omega(y) = ny[\psi(ny) - \psi(y)]. \tag{3.18}$$

Then $\omega'(y)/n = h(ny) - h(y)$, where h is defined in (3.16). Since h is strictly increasing on $(0, \infty)$, we obtain $\omega'(y) > 0$ and

$$\omega(y) > \lim_{t \to 0} \omega(t) \quad (y > 0). \tag{3.19}$$

The recurrence formula (2.5) implies

$$\lim_{t \to 0} \omega(t) = n - 1, \tag{3.20}$$

so that (3.17)–(3.20) lead to (3.14). From (3.13) and (3.14) we conclude that f is strictly decreasing on $(0, \infty)$.

If we set $x_1 = \cdots = x_n = x > 0$, then we have

$$\frac{B(\mu x_1, \dots, \mu x_n)}{B(\nu x_1, \dots, \nu x_n)} = \left(\frac{\Gamma(\mu x + 1)}{\Gamma(\nu x + 1)}\right)^n \frac{\Gamma(n\nu x + 1)}{\Gamma(n\mu x + 1)} \left(\frac{\nu}{\mu}\right)^{n-1}.$$

This implies

$$\lim_{x \to 0} \frac{B(\mu x_1, \dots, \mu x_n)}{B(\nu x_1, \dots, \nu x_n)} = \left(\frac{\nu}{\mu}\right)^{n-1}. \tag{3.21}$$

And, if we put $x_1 = x > 0$, $x_2 = \cdots = x_n = 1$, then we get

$$\frac{B(\mu x_1, \ldots, \mu x_n)}{B(\nu x_1, \ldots, \nu x_n)} = \frac{\Gamma(\mu x)}{\Gamma(\mu x + (n-1)\mu)} (\mu x)^{(n-1)\mu} \frac{\Gamma(\nu x + (n-1)\nu)}{\Gamma(\nu x)} (\nu x)^{(1-n)\nu} \times \left(\frac{\nu^\nu \Gamma(\mu)}{\mu^\mu \Gamma(\nu)} \right)^{n-1} x^{(n-1)(\nu-\mu)}. \tag{3.22}$$

From (2.4) and (3.22) we obtain

$$\lim_{x\to\infty} \frac{B(\mu x_1, \ldots, \mu x_n)}{B(\nu x_1, \ldots, \nu x_n)} = 0. \tag{3.23}$$

The limit relations (3.21) and (3.23) imply that the bounds given in (3.12) are best possible.

A function $f : \mathbf{R}_+^n \to \mathbf{R}$ is called midconvex (or Jensen-convex) if we have for all $x_i, y_i > 0$ $(i = 1, \ldots, n)$:

$$f\big((x_1 + y_1)/2, \ldots, (x_n + y_n)/2\big) \le \frac{1}{2}\big(f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n)\big). \tag{3.24}$$

It is known that a continuous midconvex function is also convex; see [17]. We now prove that $f(x_1, \ldots, x_n) = \log B(x_1, \ldots, x_n)$ satisfies (3.24), which implies that the beta function is log-convex on $\mathbf{R}_+^n$. This extends a result given in [9], where a proof for the log-convexity of $B(x, y)$ is given.

THEOREM 3.4. *Let $n \ge 2$ be an integer. Then we have for all real numbers with $x_i > 0$ and $y_i > 0$ $(i = 1, \ldots, n)$:*

$$0 < \frac{B((x_1 + y_1)/2, \ldots, (x_n + y_n)/2)}{\sqrt{B(x_1, \ldots, x_n) B(y_1, \ldots, y_n)}} \le 1. \tag{3.25}$$

Both bounds are best possible.

PROOF. The Cauchy-Schwarz inequality for integrals yields

$$\begin{aligned}\big(B(x_1 + y_1, x_2 + y_2)\big)^2 &= \left(\int_0^1 t^{x_1-1/2}(1-t)^{x_2-1/2}\, t^{y_1-1/2}(1-t)^{y_2-1/2}\, dt\right)^2 \\ &\le \int_0^1 t^{2x_1-1}(1-t)^{2x_2-1}\, dt \int_0^1 t^{2y_1-1}(1-t)^{2y_2-1}\, dt \\ &= B(2x_1, 2x_2) B(2y_1, 2y_2).\end{aligned} \tag{3.26}$$

Using the representation

$$B(x_1, \ldots, x_n) = \prod_{i=1}^{n-1} B\left(\sum_{j=1}^{i} x_j, x_{i+1}\right)$$

and (3.26) we obtain

$$\begin{aligned}\big(B(x_1+y_1,\ldots,x_n+y_n)\big)^2 &= \prod_{i=1}^{n-1}\left[B\left(\sum_{j=1}^{i}x_j+\sum_{j=1}^{i}y_j,x_{i+1}+y_{i+1}\right)\right]^2\\ &\le \prod_{i=1}^{n-1}\left[B\left(2\sum_{j=1}^{i}x_j,2x_{i+1}\right)B\left(2\sum_{j=1}^{i}y_j,2y_{i+1}\right)\right]\\ &= B(2x_1,\ldots,2x_n)B(2y_1,\ldots,2y_n).\end{aligned}$$

This proves the right-hand side of (3.25). If we set $x_i=y_i=z>0$ $(i=1,\ldots,n)$, then equality holds in the second inequality of (3.25). Further, we have

$$\lim_{x_1\to 0}\frac{\big(B((x_1+y_1)/2,\ldots,(x_n+y_n)/2)\big)^2}{B(x_1,\ldots,x_n)B(y_1,\ldots,y_n)}=0,$$

so that in (3.25) the lower bound 0 cannot be improved.

A function $f:\mathbf{R}_+^n\to\mathbf{R}$ is said to be subadditive if the inequality

$$f(x_1+y_1,\ldots,x_n+y_n)\le f(x_1,\ldots,x_n)+f(y_1,\ldots,y_n)\tag{3.27}$$

holds for all $x_i,y_i>0$ $(i=1,\ldots,n)$. Subadditive functions play a role in the theory of differential equations, in the theory of convex bodies, and also in the theory of semi-groups; see [18]. From the following theorem we conclude that for all real numbers $c>0$ the function $(x_1,\ldots,x_n)\mapsto(B(x_1,\ldots,x_n))^c$ is subadditive on $\mathbf{R}_+^n$.

THEOREM 3.5. *Let $c>0$ be a real number and let $n\ge 2$ be an integer. Then we have for all real numbers $x_i>0$ and $y_i>0$ $(i=1,\ldots,n)$:*

$$0<\frac{(B(x_1+y_1,\ldots,x_n+y_n))^c}{(B(x_1,\ldots,x_n))^c+(B(y_1,\ldots,y_n))^c}<2^{-c(n-1)-1}.\tag{3.28}$$

Both bounds are best possible.

PROOF. To prove the second inequality of (3.28) we apply Theorem 3.4, the arithmetic mean-geometric mean inequality, and Theorem 3.3 (with $\mu=2$, $\nu=1$). Then we get

$$\begin{aligned}(B(x_1+y_1,\ldots,x_n+y_n))^c &\le [B(2x_1,\ldots,2x_n)B(2y_1,\ldots,2y_n)]^{c/2}\\ &\le \frac{1}{2}[(B(2x_1,\ldots,2x_n))^c+(B(2y_1,\ldots,2y_n))^c]\\ &< 2^{-c(n-1)-1}[(B(x_1,\ldots,x_n))^c+(B(y_1,\ldots,y_n))^c].\end{aligned}$$

It remains to show that the bounds given in (3.28) are sharp. First, we set $x_i = y_i = z > 0$ $(i = 1, \dots, n)$. The duplication formula (2.3) leads to, say,

$$\frac{(B(x_1+y_1,\dots,x_n+y_n))^c}{(B(x_1,\dots,x_n))^c+(B(y_1,\dots,y_n))^c} = \frac{1}{2}\left(\frac{1}{2\sqrt{\pi}}\right)^{c(n-1)}\left(\frac{(\Gamma(z+1/2))^n}{\Gamma(nz+1/2)}\right)^c = f(z).$$

Since $\Gamma(1/2) = \sqrt{\pi}$, we obtain

$$\lim_{z\to 0} f(z) = 2^{-c(n-1)-1}. \tag{3.29}$$

And using (2.1) we get

$$\lim_{z\to \infty} f(z) = 0. \tag{3.30}$$

From (3.29) and (3.30) we conclude that both bounds in (3.28) are best possible.

REMARK. A multiplicative analogue of the definition (3.27) is given by

$$f(x_1y_1,\dots,x_ny_n) \le f(x_1,\dots,x_n)\, f(y_1,\dots,y_n). \tag{3.31}$$

If f satisfies (3.31) for all $x_i, y_i > 0$ $(i = 1, \dots, n)$, then f is said to be submultiplicative on $\mathbf{R}_+^n$. These functions have applications in functional analysis and group theory; see [11, 14]. If (3.31) holds with "$\ge$" instead of "$\le$", then f is called supermultiplicative. Let $n \ge 2$. We set $x_i = 1$ $(2 \le i \le n)$ and $y_i = 1$ $(1 \le i \le n;\ i \ne 2)$. Then we obtain, say,

$$\frac{B(x_1y_1,\dots,x_ny_n)}{B(x_1,\dots,x_n)B(y_1,\dots,y_n)} = \frac{\Gamma(x_1+n-1)}{\Gamma(x_1+y_2+n-2)} x_1^{y_2-1}\Gamma(y_2+n-1)x_1^{1-y_2}$$
$$= \sigma(x_1).$$

Applying (2.4) we get: if $y_2 > 1$, then $\lim_{x_1\to\infty}\sigma(x_1) = 0$; and, if $y_2 \in (0, 1)$, then we have $\lim_{x_1\to\infty}\sigma(x_1) = \infty$. This implies that $(x_1, \dots, x_n) \mapsto B(x_1, \dots, x_n)$ is neither submultiplicative nor supermultiplicative on $\mathbf{R}_+^n$.

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