ANZIAM J. 44(2003), 355–364

A QUASI-TRAPEZOID INEQUALITY FOR DOUBLE INTEGRALS

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(Received 22 January 1999; revised 6 October 2000)

Abstract

A quasi-trapezoid inequality is derived for double integrals that strengthens considerably existing results. A consonant version of the Grüss inequality is also derived. Applications are made to cubature formulæ and the error variance of a stationary variogram.

1. Introduction

Although important for applications, numerical integration in two or more dimensions is still a much less developed area than its one-dimensional counterpart, which has been worked on intensively. For some interesting recent commentary, see Sloan [8]. Even the traditional integration of polynomial forms over rectilinear regions translates in higher dimensions to problems with some complications (*cf*. Rathod and Govinda Rao [6]).

Central to questions of numerical integration in one dimension are Ostrowski's theorem and inequalities of trapezoid type. For a compendious treatment of the latter see Mitrinović *et al*. [5] and the references therein. Recently new versions of some of the classical tools have been developed for a two-dimensional context.

Suppose $f(\cdot,\cdot)$ is integrable on $[a,b]\times[c,d]$ and for $x\in[a,b]$ and $y\in[c,d]$ set

$$f^{\dagger}(x,y) := \int_a^b\int_c^d f(s,t)\,ds\,dt + (b-a)(d-c)f(x,y) - (b-a)\int_c^d f(x,t)\,dt - (d-c)\int_a^b f(s,y)\,ds.$$

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Barnett and Dragomir [1] have proved the following two-dimensional theorem of Ostrowski type.

THEOREM A. *If* $f(\cdot,\cdot)$ *is continuous on* $[a,b]\times[c,d]$ *and* $f''_{x,y} = \partial^2 f/\partial x\partial y$ *exists on* $(a,b)\times(c,d)$ *and is bounded, that is,*

$$\left\| f''_{s,t}\right\|_\infty := \sup_{(x,y)\in(a,b)\times(c,d)} \left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right| < \infty,$$

then for any $x\in[a,b]$ *and* $y\in[c,d]$

$$\left|f^\dagger(x,y)\right| \le \left[\frac{(b-a)^2}{4}+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{(d-c)^2}{4}+\left(y-\frac{c+d}{2}\right)^2\right]\left\| f''_{s,t}\right\|_\infty . \quad (1.1)$$

Here and subsequently it is implicit that $f''_{s,t}$ is integrable on $[a,b]\times[c,d]$.

An interesting particular case, which is in fact the best inequality we can obtain from (1.1), is the 'quasi-midpoint' inequality

$$\left|f^\dagger((a+b)/2,(c+d)/2)\right| \le (1/16)(b-a)^2(d-c)^2\left\| f''_{s,t}\right\|_\infty .$$

The first two authors have applied (1.1) to cubature formulæ in [1] and to the analysis of variograms in [2].

Define the functional

$$\begin{aligned} f^* &:= \left[f^\dagger(a,c)+f^\dagger(a,d)+f^\dagger(b,c)+f^\dagger(b,d)\right]/4 \\ &= \int_a^b\int_c^d f(s,t)\,dt\,ds + \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c) \\ &\quad -(d-c)\int_a^b \frac{f(s,c)+f(s,d)}{2}\,ds - (b-a)\int_c^d \frac{f(a,t)+f(b,t)}{2}\,dt. \end{aligned}$$

When Theorem A applies, we have

$$|f^\dagger(a,c)| \le (1/4)(b-a)^2(d-c)^2\left\| f''_{s,t}\right\|_\infty$$

and similarly for $f^\dagger(a,d)$, $f^\dagger(b,c)$ and $f^\dagger(b,d)$, so that

$$|f^*| \le (1/4)(b-a)^2(d-c)^2\left\| f''_{s,t}\right\|_\infty .$$

In this article we show that a much stronger result holds, namely the following.

THEOREM 1. *Under the conditions of Theorem A,*

$$|f^*| \le \frac{1}{16}(b-a)^2(d-c)^2\left\| f''_{s,t}\right\|_\infty .$$

This we establish in Section 2, where it is shown that it follows from an appropriate double-integral identity. In Section 3 we derive a conformable inequality of Grüss type and in Section 4 apply our ideas to cubature formulæ. We conclude in Section 5 with an application to bounds on the error variance of a continuous stream with stationary variogram.

2. Integral identities

First we derive a useful ancillary result.

LEMMA 1. *Suppose that* $\alpha_1 < \alpha_2$ *and* $\beta_1 < \beta_2$ *and that* $\partial^2 f/\partial s \partial t$ *is integrable on* $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$. *If either* $(\alpha, \alpha') = (\alpha_1, \alpha_2)$ *or* $(\alpha', \alpha) = (\alpha_1, \alpha_2)$ *and similarly for* β, β', *then*

$$\begin{aligned}\int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} (s-\alpha)(t-\beta) f''_{s,t}\, dt\, ds &\\ = (\alpha_2-\alpha_1)(\beta_2-\beta_1) f(\alpha', \beta') &- (\beta_2-\beta_1)\int_{\alpha_1}^{\alpha_2} f(s, \beta')\, ds\\ - (\alpha_2-\alpha_1)\int_{\beta_1}^{\beta_2} f(\alpha', t)\, dt &+ \int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2} f(s,t)\, dt\, ds.\end{aligned}$$

PROOF. This is immediate from a repeated integration by parts.

We now proceed to our main double-integral identity.

THEOREM 2. *Under the assumptions of Theorem A,*

$$f^* = \int_a^b \int_c^d \left(s - \frac{a+b}{2}\right)\left(t - \frac{c+d}{2}\right) f''_{s,t}(s,t)\, dt\, ds. \tag{2.1}$$

PROOF. Take $x \in [a, b]$, $y \in [c, d]$ and apply Lemma 1 with the four choices

$$\begin{aligned}&(\alpha_1, \alpha_2, \beta_1, \beta_2, \alpha, \beta)\\ &\quad = (a, x, c, y, a, c), (a, x, y, d, a, d), (x, b, y, d, b, d), (x, b, c, y, b, c).\end{aligned}$$

Addition of the resultant identities yields

$$\begin{aligned}\int_a^b \int_c^d p(x,s) q(y,t) f''_{s,t}\, dt\, ds &\\ = (d-c)(b-a) f(x,y) &- (d-c)\int_a^b f(s,y)\, ds\\ - (b-a)\int_c^d f(x,t)\, dt &+ \int_a^b \int_c^d f(s,t)\, dt\, ds,\end{aligned}$$

where $p(x, s)$ is defined as $s-a$ if $s \in [a, x]$ and as $s-b$ if $s \in (x, b]$, whilst $q(y, t)$ is $t-c$ if $t \in [c, y]$ and $t-d$ if $t \in (y, d]$.

We now make the four choices

$$(x, y) = (a, c), (b, c), (a, d), (b, d)$$

and add again to derive

$$\begin{aligned}
&\int_a^b \int_c^d [p(a,s)+p(b,s)][q(c,t)+q(d,t)]\, f''_{s,t}(s,t)\,dt\,ds \\
&\quad = 4\int_a^b \int_c^d f(s,t)\,dt\,ds \\
&\qquad + [f(a,c)+f(a,d)+f(b,c)+f(b,d)](b-a)(d-c) \\
&\qquad - 2(d-c)\int_a^b [f(s,c)+f(s,d)]\,ds - 2(b-a)\int_c^d [f(a,t)+f(b,t)]\,dt.
\end{aligned}$$

Since

$$p(a,s)+p(b,s) = 2s-(a+b), \qquad q(c,t)+q(d,t) = 2t-(c+d),$$

this is equivalent to the desired identity.

Our theorem provides

$$|f^*| \le \int_a^b \int_c^d \left|s-\frac{a+b}{2}\right| \left|t-\frac{c+d}{2}\right| f''_{s,t}(s,t)\,dt\,ds$$

and a simple calculation yields

$$\int_\alpha^\beta \left|u-\frac{\alpha+\beta}{2}\right| du = \frac{(\beta-\alpha)^2}{4}. \tag{2.2}$$

Theorem 1 follows as an immediate corollary.

3. An inequality of Grüss type

The well-known Grüss inequality (see for example Mitrinović *et al*. [4, p. 296]) states that if $f, g : [a,b] \to \mathbf{R}$ are integrable on $[a,b]$ and

$$\varphi \le f(x) \le \Phi, \qquad \gamma \le g(x) \le \Gamma \quad \text{for all } s \in [a,b],$$

then

$$|I| \le \frac{1}{4}(b-a)^2(\Gamma-\gamma)(\Phi-\varphi),$$

where

$$I := (b-a)\int_a^b f(x)g(x)\,dx - \int_a^b f(x)\,dx \int_a^b g(x)\,dx.$$

Moreover, the constant $1/4$ is best possible.

We establish a closely related result.

THEOREM 3. *Suppose* $f, g : [a, b] \to \mathbf{R}$ *are continuous on* $[a, b]$*, differentiable on* (a, b) *and with bounded derivatives. Put*

$$\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty, \qquad \|g'\|_\infty := \sup_{t\in(a,b)} |g'(t)| < \infty.$$

Then

$$\begin{aligned} &\left|I + [f(a) - f(b)][g(a) - g(b)](b-a)^2/4\right| \\ &\quad \le \frac{(b-a)^2}{2}\left[\|f - f(a)\|_\infty \|g - g(a)\|_\infty + \|f - f(b)\|_\infty \|g - g(b)\|_\infty\right] \\ &\qquad + \frac{(b-a)^4}{16} \|f'\|_\infty \|g'\|_\infty . \end{aligned} \tag{3.1}$$

PROOF. Define $h : [a, b]^2 \to \mathbf{R}$ by $h(s, t) = [f(s) - f(t)][g(s) - g(t)]$. We have

$$h(a, a) + h(a, b) + h(b, a) + h(b, b) = 2[f(b) - f(a)][g(b) - g(a)]$$

and

$$\begin{aligned} &\int_a^b [h(s, a) + h(s, b)]\, ds \\ &\quad = \int_a^b [h(a, s) + h(b, s)]\, ds \\ &\quad = \int_a^b \left\{[f(s) - f(a)][g(s) - g(a)] + [f(b) - f(s)][g(b) - g(s)]\right\} ds. \end{aligned}$$

Also

$$\frac{\partial^2 h(s,t)}{\partial s \partial t} = -f'(s)g'(t) - f'(t)g'(s),$$

so that

$$\begin{aligned} &\int_a^b \int_a^b \left(s - \frac{a+b}{2}\right)\left(t - \frac{a+b}{2}\right) \frac{\partial^2 h(s,t)}{\partial s \partial t}\, dt\, ds \\ &\quad = -2\int_a^b \left(s - \frac{a+b}{2}\right) f'(s)\, ds \int_a^b \left(t - \frac{a+b}{2}\right) g'(t)\, dt. \end{aligned}$$

Hence applying Theorem 2 to h on $[a, b] \times [a, b]$ provides

$$\begin{aligned} &\int_a^b \int_a^b [f(s) - f(t)][g(s) - g(t)]\, ds\, dt + \frac{[f(b) - f(a)][g(b) - g(a)]}{2} (b-a)^2 \\ &\quad = (b-a)\int_a^b \left\{[f(s) - f(a)][g(s) - g(a)] + [f(b) - f(s)][g(b) - g(s)]\right\} ds \\ &\qquad - 2\int_a^b \left(s - \frac{a+b}{2}\right) f'(s)\, ds \int_a^b \left(t - \frac{a+b}{2}\right) g'(t)\, dt. \end{aligned}$$

Since

$$\frac{1}{2}\int_a^b\int_a^b [f(s)-f(t)][g(s)-g(t)]\,ds\,dt = I,$$

we deduce that

$$\begin{aligned}I &+ \frac{[f(b)-f(a)][g(b)-g(a)]}{4}(b-a)^2\\ &= \frac{b-a}{2}\int_a^b \{[f(s)-f(a)][g(s)-g(a)]+[f(b)-f(s)][g(b)-g(s)]\}\,ds\\ &\quad -\int_a^b\left(s-\frac{a+b}{2}\right)f'(s)\,ds\int_a^b\left(t-\frac{a+b}{2}\right)g'(t)\,dt.\end{aligned}$$

Thus the left-hand side of (3.1) is bounded above by

$$\begin{aligned}&\frac{1}{2}(b-a)^2\big[\|f-f(a)\|_\infty\|g-g(a)\|_\infty+\|f(b)-f\|_\infty\|g(b)-g\|_\infty\big]\\ &\quad+\|f'\|_\infty\|g'\|_\infty\left[\int_a^b\left|s-\frac{a+b}{2}\right|ds\right]^2.\end{aligned}$$

The desired result follows from (2.2).

4. Application to cubature formulæ

Take arbitrary divisions $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of $[a, b]$ and $J_m : c = y_0 < y_1 < \cdots < y_{m-1} < y_m = d$ of $[c, d]$ and set $h_i := x_{i+1} - x_i$ $(i = 0, \ldots, n-1)$ and $l_j := y_{j+1} - y_j$ $(j = 0, \ldots, m-1)$. Define

$$\begin{aligned}\eta_{i,j} := h_i\int_{y_j}^{y_{j+1}} f\left(\frac{x_i+x_{i+1}}{2}, t\right)dt + l_j\int_{x_i}^{x_{i+1}} f\left(s, \frac{y_j+y_{j+1}}{2}\right)ds\\ - h_i l_j\, f\left(\frac{x_i+x_{i+1}}{2}, \frac{y_j+y_{j+1}}{2}\right).\end{aligned}$$

Barnett and Dragomir [1] considered a quasi-midpoint rule for double integrals given by

$$C_M(f, I_n, I_m) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\eta_{i,j}$$

and proved that provided the integrals involved exist and $\|f''_{s,t}\|_\infty$ is finite, then

$$\int_a^b\int_c^d f(s,t)\,ds\,dt = C_M(f, I_n, J_m) + R_M(f, I_n, J_m),$$

where the remainder satisfies

$$|R_M(f, I_n, J_m)| \le \frac{1}{16} \| f''_{s,t} \|_\infty \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2.$$

We are now able to establish a quasi-trapezoid formula. Set

$$\xi_{i,j} := h_i \int_{y_j}^{y_{j+1}} \left[\frac{f(x_i, t) + f(x_{i+1}, t)}{2} \right] dt + l_j \int_{x_j}^{x_{j+1}} \left[\frac{f(s, y_j) + f(s, y_{j+1})}{2} \right] ds$$
$$- h_i l_j \left[\frac{f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})}{4} \right]$$

and define

$$C_T(f, I_n, J_m) := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \xi_{i,j}.$$

Then we have the following result.

THEOREM 4. *Let $f : [a, b] \times [c, d] \to \mathbf{R}$ satisfy the conditions of Theorem A. Then we have the cubature formula*

$$\int_a^b \int_c^d f(s, t)\, ds\, dt = C_T(f, I_n, J_m) + R_T(f, I_n, J_m),$$

where the remainder term satisfies

$$|R_T(f, I_n, J_m)| \le \frac{1}{16} \| f''_{s,t} \|_\infty \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2. \tag{4.1}$$

PROOF. Applying Theorem 1 to the interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ for $i = 0, \dots, n-1$ and $j = 0, \dots, m-1$ gives

$$\left| \int_{xi}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(s, t)\, dt\, ds - \xi_{i,j} \right| \le \frac{1}{16} h_i^2 l_j^2 \| f''_{s,t} \|_\infty .$$

Summing over i from 0 to $n-1$ and j from 0 to $m-1$ and using the generalized triangle inequality yields the desired inequality (4.1).

REMARK 1. Set

$$\nu(h) := \max\{h_i : i = 0, \dots, n-1\}, \quad \mu(l) := \max\left\{l_j := j = 0, \dots, m-1\right\}.$$

Then since

$$\sum_{i=0}^{n-1} h_i^2 \le \nu(h) \sum_{i=0}^{n-1} h_i = (b-a)\nu(h)$$

and

$$\sum_{j=0}^{m-1} l_j^2 \le \mu(l) \sum_{j=0}^{m-1} l_j = (d-c)\mu(l),$$

the right-hand side of (4.1) is bounded above by

$$\frac{1}{16} \left\| f_{s,t}'' \right\|_\infty (b-a)\,(d-c)\nu(h)\mu(l),$$

which is of order two precision.

5. The error variance of a continuous stream with stationary variogram

Suppose $(X(t))$ is a continuous-time stochastic process, possibly nonstationary. Typically $(X(t))$ represents a continuous-stream industrial process such as is common in many areas of the chemical industry. In [3], the authors considered $X(t)$ as defining the quality of a product at time t. The paper was concerned with issues related to sampling the stream with a view to estimating the mean quality $\bar{X}$ characteristic of the flow over the interval $[0, d]$. The sampling location t is said to be optimal if it minimizes the estimation error variance

$$E\left[\left(\bar{X} - X(t)\right)^2\right], \qquad 0 < t < d.$$

In [3] it was shown that for constant stream flows, the optimal sampling point is the midpoint of $[0, d]$ for the situation where the process variogram

$$V(u) = \frac{1}{2} E\left[(X(t) - X(t+u))^2\right],$$
$$V(0) = 0, \quad V(-u) = V(u), \quad u \in [-d, d]$$

is stationary. We remark that variogram stationarity is not equivalent to process stationarity.

In this paper we use Theorem 1 to give an approximation of the estimation error variance $E[(\bar{X} - X(t))^2]$ for $t = d$.

From [3], it can be shown using an identity given in [7] that

$$E\left[\left(\bar{X} - X(t)\right)^2\right] = -\frac{1}{d^2} \int_0^d \int_0^d V(v-u)\,du\,dv + \frac{2}{d}\left\{\int_0^t V(u)\,du + \int_0^{d-t} V(u)\,du\right\}. \tag{5.1}$$

Suppose V is continuous on $[-d, d]$, twice differentiable on $(-d, d)$ and has bounded second derivative V'' bounded on that interval. It is shown in [2] that from

(1.1) it is possible to get the bound

$$E\left[\left(\bar{X}-X(t)\right)^2\right] \leq \left[\frac{1}{4}+\frac{(t-d/2)^2}{d^2}\right]^2 d^2 \left\|V''\right\|_\infty \tag{5.2}$$

for all $t \in [0, d]$.

The best inequality we can get from (5.2) is for $t = t_0 = d/2$ when we have the bound

$$E\left[\left(\bar{X}-X(d/2)\right)^2\right] \leq \frac{d^2}{16}\left\|V''\right\|_\infty .$$

For $t = d$,

$$E\left[\left(\bar{X}-X(d)\right)^2\right] \leq \frac{d^2}{4}\left\|V''\right\|_\infty .$$

This can be complemented as follows.

Put $f(s,t) = V(s-t)$, $a = c = 0$ and $b = d$ in Theorem 1 to get

$$\left|\int_0^d \int_0^d V(s-t)\,ds\,dt + \frac{V(0)+V(-d)+V(d)+V(0)}{4}\,d^2 \right.$$
$$\left. -d\int_0^d \frac{V(-t)+V(d-t)}{2}\,dt - d\int_0^d \frac{V(s)+V(s-d)}{2}\,ds\right| \leq \frac{d^4}{16}\left\|V''\right\|_\infty . \tag{5.3}$$

Since $V(0) = 0$ and $V(-d) = V(d)$, we have

$$\int_0^d \frac{V(-t)+V(d-t)}{2}\,dt = \int_0^d \frac{V(s)+V(s-d)}{2}\,ds$$
$$= \int_0^d \frac{V(t)+V(d-t)}{2}\,dt = \int_0^d V(u)\,du$$

and by (5.3)

$$\left|\int_0^d \int_0^d V(s-t)\,ds\,dt - 2d\int_0^d V(u)\,du + \frac{V(d)}{2}\,d^2\right| \leq \frac{d^4}{16}\left\|V''\right\|_\infty . \tag{5.4}$$

But, by the identity (5.1), we deduce that

$$\int_0^d \int_0^d V(s-t)\,ds\,dt - 2d\int_0^d V(u)\,du = -d^2 E\left[\left(\bar{X}-X(d)\right)^2\right].$$

Consequently, by (5.4), we get

$$\left|E\left[\left(\bar{X}-X(d)\right)^2\right] - \frac{V(d)}{2}\right| \leq \frac{d^2}{16}\left\|V''\right\|_\infty$$

which gives an approximation for $E[(\bar{X}-X(d))^2]$ in terms of $V(d)$.

Note that for small d the approximation is accurate and is of order two precision.

References

[1] N. S. Barnett and S. S. Dragomir, "An Ostrowski type inequality for double integrals and applications to cubature formulae", *Soochow J. Math.*, to appear.

[2] N. S. Barnett and S. S. Dragomir, "A note on bounds for the estimation error variance of a continuous stream with stationary variogram", *J. KSIAM* **2** (1998) 49–56.

[3] N. S. Barnett, I. S. Gomm and L. Armour, "Location of the optimal sampling point for the quality assessment of continuous streams", *Austral. J. Stat.* **37** (1995) 145–152.

[4] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and new inequalities in analysis* (Kluwer, Dordrecht, 1993).

[5] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities for functions and their integrals and derivatives* (Kluwer, Dordrecht, 1994).

[6] H. T. Rathod and H. S. Govinda Rao, "Integration of trivariate polynomials over linear polyhedra in Euclidean three-dimensional space", *J. Austral. Math. Soc. Ser. B* **39** (1998) 355–385.

[7] I. W. Saunder, G. K. Robinson, T. Lwin and R. J. Holmes, "A simplified variogram method for the estimation error variance in sampling from continuous stream", *Internat. J. Mineral Processing* **25** (1989) 175–198.

[8] I. H. Sloan, "Multiple integration is intractable but not hopeless", *ANZIAM J.* **42** (2000) 3–8.