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EXPONENTIAL DECAY RATE OF THE ENERGY OF A TIMOSHENKO BEAM WITH LOCALLY DISTRIBUTED FEEDBACK

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Abstract

The problem of the energy exponential decay rate of a Timoshenko beam with locally distributed controls is investigated. Consider the case in which the beam is nonuniform and the two wave speeds are different. Then, using Huang's theorem and Birkhoff's asymptotic expansion method, it is shown that, under some locally distributed controls, the energy exponential decay rate is identical to the supremum of the real part of the spectrum of the closed loop system. Furthermore, explicit asymptotic locations of eigenfrequencies are derived.

1. Introduction

The main objective of this paper is to investigate the energy exponential decay rate of a beam with locally distributed feedback controls. It is well-known that if the cross-section dimensions of a beam are not negligible compared with its length, then it is necessary to consider the effect of the rotational inertia. Furthermore, if the deflection due to shear is also not negligible, then the beam is called a Timoshenko beam and can be described by (see [17]):

$$\begin{cases} \rho \dfrac{\partial^2 w}{\partial t^2} - \dfrac{\partial}{\partial x}\left(K \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial x}(K\varphi) + u_1(x,t) = 0, & 0 < x < \ell,\ t > 0, \\ I_\rho \dfrac{\partial^2 \varphi}{\partial t^2} - \dfrac{\partial}{\partial x}\left(EI \dfrac{\partial \varphi}{\partial x}\right) + K\left(\varphi - \dfrac{\partial w}{\partial x}\right) + u_2(x,t) = 0, & 0 < x < \ell,\ t > 0. \end{cases} \tag{1}$$

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Here a nonuniform beam of length ℓ moves in the $(w$–$x)$-plane, $\rho(x)$ is the mass density, $w(x,t)$ is the deflection of the beam from its equilibrium, and $\varphi(x,t)$ is the total rotational angle of the beam at x. For precise physical details, see [17]. The terms $I_\rho(x)$ and $EI(x)$ are the mass moment of inertia and rigidity coefficient of the cross section, respectively, and $K(x)$ is the shear modulus of elasticity. Also $u_1(x,t)$ and $u_2(x,t)$ are locally distributed controls, that is, there exists a subinterval $[\alpha, \beta] \subset [0, \ell]$, such that

$$u_1(x,t) \equiv 0, \quad u_2(x,t) \equiv 0, \qquad x \notin [\alpha, \beta],\ t \geq 0.$$

In the case of a cantilever configuration, the appropriate boundary conditions are

$$\begin{cases} w(0,t) = 0, \quad \varphi(0,t) = 0, \\ K(\ell)[\varphi(\ell,t) - w'(\ell,t)] = \mu_1(t), \\ -EI(\ell)\varphi'(\ell,t) = \mu_2(t), \end{cases} \tag{2}$$

where $\mu_1(t)$ and $\mu_2(t)$ are the applied lateral force and moment at $x = \ell$, respectively. To avoid lengthy calculations, which are not necessary for our purpose, one can assume that $\mu_1(t) = \mu_2(t) \equiv 0$. Here and below, the prime and the dot are used to denote derivatives with respect to space and time variables, respectively.

In recent years, control problems involving large flexible structures have attracted much attention, for example, see [3, 4, 6]. In [9], the boundary feedback control of a Timoshenko beam was considered. In [5], the locally distributed control of a uniform Euler-Bernoulli beam was studied. Recently, the use of "smart material" as sensors and actuators has burgeoned in applications (for more information, refer to [1]). When the parts made of "smart materials" are boned or embedded to the underlying structure as locally distributed damping in stabilizing the vibration of the flexible structure, in order to obtain an optimal allocation result, it is necessary to know whether the energy decay rate of the system is identical to the supremum of the real part of the spectrum of the closed loop system. The purpose of this paper is to establish the exponential stabilization of Timoshenko beam vibration by using locally distributed feedback controls and to find the energy decay rate. As shown above, the vibration of a Timoshenko beam is described by two coupled wave equations with variable coefficients. Although the multiplier methods used in [7, 11] are sufficient to prove the exponential stability of a nonhomogeneous one-dimensional elastic system, they cannot provide exact information for the energy exponential decay rate of the system. On the other hand, the methods proposed in [10, 14] also have deficiencies, as it is very difficult to know whether or not the eigenvalues of the system satisfy the gap condition. This is a usual (although not a necessary) condition for establishing the stabilization of elastic vibration.

Huang's theorem (see [8]) gives a necessary and sufficient condition for the validity of the spectrum-determined growth assumption of an infinite-dimensional system. This condition has been used by many authors to investigate the exponential stability of a system with constant coefficients (see, for example, [12]). However, it is difficult to verify this condition for a Timoshenko beam system with variable coefficients because no explicit formula for the resolvent is available. We use Birkhoff's asymptotic expansion method introduced in [2] to estimate the norm of the resolvent operator. The same method is used in [18] to estimate the locations of eigenfrequencies. We note that the form of the asymptotic solutions obtained in this paper appears to be rather different to that obtained in [18], because the first-order equation with a large parameter is non-homogeneous. In Section 2, some locally distributed feedback controls are proposed, and the well-posedness of the corresponding closed loop system is given *via* semigroup theory. We consider the case in which the wave speeds are different (this condition is, in general, satisfied, see [15]). The asymptotic solutions of the resolvent equation are given by using Birkhoff's asymptotic expansion method. In Section 3, asymptotic estimates of eigenvalues are derived using Rouche's theorem. Finally in Section 4, it is shown that the spectrum-determined growth assumption holds, and that the vibration of the beam decays exponentially by virtue of Huang's theorem. Thus the energy exponential decay rate is identical to the supremum of the real part of the spectrum of the closed loop system.

2. Preliminary results

We propose the following locally distributed feedback controls:

$$u_1(x,t) = \rho(x)b_1(x)\dot{w}(x,t), \quad u_2(x,t) = I_\rho(x)b_2(x)\dot{\varphi}(x,t), \tag{3}$$

where $b_j(x) \in C^1[0,\ell]$ for $j = 1, 2$, and

$$\begin{cases} b_j(x) \equiv 0, & x \notin [\alpha,\beta] \\ b_j(x) \geq 0, & x \in [\alpha,\beta] \\ b_j(x) \geq \gamma > 0, & x \in [c,d] \subset [\alpha,\beta]. \end{cases} \tag{4}$$

Then the closed loop system corresponding to (1) and (2) becomes

$$\begin{cases} \rho\ddot{w} - (Kw')' + (K\varphi)' + \rho b_1\dot{w}(x,t) = 0, & 0 < x < \ell,\ t > 0, \\ I_\rho\ddot{\varphi} - (EI\varphi')' + K(\varphi - w') + I_\rho b_2\dot{\varphi}(x,t) = 0, & 0 < x < \ell,\ t > 0. \\ w(0,t) = 0, \quad \varphi(0,t) = 0, \\ K(\ell)[\varphi(\ell,t) - w'(\ell,t)] = 0, \quad EI(\ell)\varphi'(\ell,t) = 0. \end{cases} \tag{5}$$

Now we incorporate the closed loop system (5) into a certain function space. To this end, we define the product Hilbert space $\mathscr{H} = V_0^1 \times L^2_\rho(0,\ell) \times V_0^1 \times L^2_{I_\rho}(0,\ell)$, where

$$V_0^k = \left\{\varphi \in H^k(0,\ell) \mid \varphi(0) = 0\right\}, \quad k = 1, 2,$$

and $H^k(0,\ell)$ is the usual Sobolev space of order k. The inner product in $\mathscr{H}$ is defined as follows:

$$\begin{aligned}(Y_1, Y_2)_{\mathscr{H}} = \int_0^\ell K(\varphi_1 - w_1')(\varphi_2 - w_2')\,dx + \int_0^\ell EI\varphi_1'\varphi_2'\,dx \\ + \int_0^\ell \rho z_1 z_2\,dx + \int_0^\ell I_\rho \psi_1\psi_2\,dx,\end{aligned}$$

where $Y_k = [w_k, z_k, \varphi_k, \psi_k]^\tau \in \mathscr{H}$ for $k = 1, 2$. We define linear operators $\mathscr{A}$ and $\mathscr{B}$ in $\mathscr{H}$:

$$\mathscr{A}\begin{bmatrix} w \\ z \\ \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} z \\ -(K(\varphi - w'))'/\rho \\ \psi \\ (EI\varphi')'/I_\rho - K(\varphi - w')/I_\rho \end{bmatrix}, \qquad \begin{bmatrix} w \\ z \\ \varphi \\ \psi \end{bmatrix} \in \mathscr{D}(\mathscr{A}),$$

$$\mathscr{D}(\mathscr{A}) = \left\{[w, z, \varphi, \psi]^\tau \in \mathscr{H} \mid w, \varphi \in V_0^2,\ z, \psi \in V_0^1,\ \varphi(\ell) = w'(\ell),\ \varphi'(\ell) = 0\right\},$$

$$\mathscr{B}\begin{bmatrix} w \\ z \\ \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ -b_1 z \\ 0 \\ -b_2\psi \end{bmatrix}, \qquad \begin{bmatrix} w \\ z \\ \varphi \\ \psi \end{bmatrix} \in \mathscr{D}(\mathscr{B}) = \mathscr{H}.$$

Let $\mathscr{A}_1 = \mathscr{A} + \mathscr{B}$. Then we can write the closed loop system (5) as a linear evolution equation in $\mathscr{H}$:

$$\dot{Y}(t) = \mathscr{A}_1 Y(t), \tag{6}$$

where $Y(t) = [w(\cdot,t), \dot{w}(\cdot,t), \varphi(\cdot,t), \dot{\varphi}(\cdot,t)]^\tau$. The energy corresponding to the solution of the closed loop system (5) is

$$\begin{aligned}E(t) = \frac{1}{2}\|Y(t)\|^2_{\mathscr{H}} = \frac{1}{2}\bigg[\int_0^\ell EI|\varphi'|^2\,dx + \int_0^\ell K|\varphi - w'|^2\,dx \\ + \int_0^\ell \rho|\dot{w}|^2\,dx + \int_0^\ell I_\rho|\dot{\varphi}|^2\,dx\bigg],\end{aligned}$$

where $Y(t) = [w(\cdot,t), \dot{w}(\cdot,t), \varphi(\cdot,t), \dot{\varphi}(\cdot,t)]^\tau$ is the solution to (6). It is easy to verify the following result.

LEMMA 1. *The linear operator $\mathscr{A}$ defined above is skew adjoint in the Hilbert space $\mathscr{H}$, and $\mathscr{A}_1$ has a compact resolvent.*

For the well-posedness of the closed loop system, we have the following theorem.

THEOREM 1. *For any initial data $Y_0 \in \mathscr{H}$, (6) has a unique weak solution $Y(t)$ such that $Y(\cdot) \in C([0,\infty),\mathscr{H})$. Moreover, if $Y_0 \in \mathscr{D}(\mathscr{A})$, then $Y(t)$ is the unique strong solution to (6) such that $Y(\cdot) \in C^1([0,\infty),\mathscr{H}) \cap C([0,\infty),\mathscr{D}(\mathscr{A}))$.*

PROOF. It follows from (4) that there exists a constant $c > 0$ such that

$$-\operatorname{Re}(\mathscr{B}Y, Y)_{\mathscr{H}} \geq c\|\mathscr{B}Y\|^2_{\mathscr{H}}, \quad \forall Y \in \mathscr{H}. \tag{7}$$

Since $\mathscr{A}$ is skew adjoint, $\operatorname{Re}(\mathscr{A}_1 Y, Y)_{\mathscr{H}} = \operatorname{Re}(\mathscr{B}Y, Y)_{\mathscr{H}} \leq 0$, which means that $\mathscr{A}_1$ is dissipative. Moreover, from Lemma 1, we know the maximal dissipativity of $\mathscr{A}_1$. According to the Lummer-Phillips theorem [13], $\mathscr{A}_1$ is an infinitesimal generator of a C_0-semigroup of contraction. Thus, by the appropriate properties of a C_0-semigroup, the desired conclusions are derived.

In general, the speeds $\sqrt{K(x)/\rho(x)}$ and $\sqrt{EI(x)/I_\rho(x)}$ of the wave are different. For details, see [15]. For simplicity, we denote by $\rho_1(x)$ and $\rho_2(x)$ the reciprocals of the two speeds $\sqrt{\rho(x)/K(x)}$ and $\sqrt{I_\rho(x)/EI(x)}$, respectively.

To obtain the asymptotic solutions of the resolvent equation using Birkhoff's asymptotic expansion method, we assume, throughout the rest of this paper, that $\rho(x)$, $K(x)$, $EI(x)$, $I_\rho(x) \in C^1[0,\ell]$ and that

$$\rho(x), K(x), EI(x), I_\rho(x) \geq \gamma_1 > 0, \quad x \in [0,\ell], \tag{8}$$

$$\rho_1(x) \neq \rho_2(x), \quad \forall x \in [0,\ell]. \tag{9}$$

First we need to transform the resolvent equation into a first-order system with a large parameter.

For any $Y_1 = [w_1, z_1, \varphi_1, \psi_1]^\tau \in \mathscr{H}$, let $Y = [w, z, \varphi, \psi]^\tau \in \mathscr{D}(\mathscr{A})$ such that $\lambda Y - \mathscr{A}_1 Y = Y_1$, that is,

$$\begin{cases} \lambda w - z = w_1, \quad (\lambda + b_1)z + (K(\varphi - w'))'/\rho = z_1, \\ \lambda\varphi - \psi = \varphi_1, \quad (\lambda + b_2)\psi - (EI\varphi')'/I_\rho + K(\varphi - w')/I_\rho = \psi_1, \\ w(0) = z(0) = \varphi(0) = \psi(0) = \varphi'(\ell) = 0, \quad \varphi(\ell) = w'(\ell). \end{cases} \tag{10}$$

Eliminating z and ψ in (10), we get the following boundary value problem on w and φ:

$$\begin{cases} w'' + \dfrac{K'}{K}w' - \dfrac{\rho}{K}\lambda(\lambda + b_1)w - \varphi' - \dfrac{K'}{K}\varphi = -\dfrac{\rho}{K}(z_1 + (\lambda + b_1)w_1), \\ \varphi'' + \dfrac{EI'}{EI}\varphi' - \dfrac{K}{EI}\varphi - \dfrac{I_\rho}{EI}\lambda(\lambda + b_2)\varphi + \dfrac{K}{EI}w' = -\dfrac{I_\rho}{EI}(\psi_1 + (\lambda + b_2)\varphi_1), \\ w(0) = \varphi(0) = \varphi'(\ell) = 0, \quad \varphi(\ell) = w'(\ell). \end{cases} \tag{11}$$

Let $u = [w, \varphi, w'/\lambda, \varphi'/\lambda]^\tau$. Then (11) is equivalent to

$$\begin{cases} u' = \lambda \begin{bmatrix} 0 & I_2 \\ A^2 & 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ B & D \end{bmatrix} u + \lambda^{-1} \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ f_1 \end{bmatrix} + \lambda^{-1} \begin{bmatrix} 0 \\ g_1 \end{bmatrix}, \\ B_1 u(0) = 0, \quad B_2 u(\ell) = 0, \end{cases} \tag{12}$$

where I_2 is the 2×2 identity matrix and $A, B, C, D, f_1, g_1, B_1, B_2$ are defined as

$$A = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}, \qquad B = \begin{bmatrix} \rho_1^2 b_1 & 0 \\ 0 & \rho_2^2 b_2 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & \frac{K'}{K} \\ 0 & \frac{K}{EI} \end{bmatrix},$$

$$D = \begin{bmatrix} -\frac{K'}{K} & 1 \\ -\frac{K}{EI} & -\frac{EI'}{EI} \end{bmatrix}, \quad f_1 = \begin{bmatrix} -\rho_1^2 w_1 \\ -\rho_2^2 \varphi_1 \end{bmatrix}, \qquad g_1 = \begin{bmatrix} -\rho_1^2 (z_1 + b_1 w_1) \\ -\rho_2^2 (\psi_1 + b_2 \varphi_1) \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To diagonalise the dominant coefficient matrix $\left[\begin{smallmatrix} 0 & I_2 \\ A^2 & 0 \end{smallmatrix}\right]$, let $v = Q^{-1} u$, where

$$Q = \frac{1}{2} \begin{bmatrix} A^{-1} & -A^{-1} \\ I_2 & I_2 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} A & I_2 \\ -A & I_2 \end{bmatrix}.$$

Then (12) becomes

$$\begin{cases} v' = \lambda M v + \tilde{B} v + \lambda^{-1} \tilde{C} v + F_1 + \lambda^{-1} G_1, \\ \tilde{B}_1 v(0) = 0, \quad \tilde{B}_2 v(\ell) = 0, \end{cases} \tag{13}$$

where

$$M = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}, \quad \tilde{B} = -Q^{-1} Q' + Q^{-1} \begin{bmatrix} 0 & 0 \\ B & D \end{bmatrix} Q, \quad \tilde{C} = Q^{-1} \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} Q,$$

$$F_1 = \begin{bmatrix} f_1 \\ f_1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} g_1 \\ g_1 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} \rho_1^{-1}(0) & 0 & -\rho_1^{-1}(0) & 0 \\ 0 & \rho_2^{-1}(0) & 0 & -\rho_2^{-1}(0) \end{bmatrix},$$

$$\tilde{B}_2 = \begin{bmatrix} -\lambda & \rho_2^{-1}(\ell) & -\lambda & -\rho_2^{-1}(\ell) \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The solution of (13) can be written as

$$v(x) = T_c(x) \Theta + v_p(x), \tag{14}$$

where $\Theta = [\theta_1, \theta_2, \theta_3, \theta_4]^\tau$ is a constant vector to be determined later, $T_c(x)$ is the fundamental solution matrix of the homogeneous equation associated with (13), and $v_p(x)$ is a particular solution of the non-homogeneous equation (13). Also $T_c(x)$ and $v_p(x)$ satisfy, respectively,

$$T_c' = (\lambda M + \tilde{B} + \lambda^{-1} \tilde{C}) T_c, \tag{15}$$

$$v_p(x) = T_c(x) \int_0^x T_c^{-1}(s) (F_1(s) + \lambda^{-1} G_1(s))\, ds. \tag{16}$$

Next, applying Birkhoff's arguments given in [2] to (15), and following a procedure similar to that given in [18], it follows that, for $\lambda \in G^- = \{\lambda \mid \operatorname{Re}\lambda \le 0, |\lambda| > N\}$ with N given large enough,

$$T_c(x) = \big(\exp(\mu_j(x))\{\delta_{kj} + \xi_{kj}(x,\lambda)/\lambda\}\big)_{4\times 4}, \tag{17}$$

$$T_c^{-1}(x) = \big(\exp(-\mu_k(x))\{\delta_{kj} + \eta_{kj}(x,\lambda)/\lambda\}\big)_{4\times 4}, \tag{18}$$

where $\mu_j(x) = \int_0^x \tilde{b}_j(s)\,ds + \lambda\int_0^x m_j(s)\,ds$, $\tilde{b}_j(s)$ and $m_j(s)$ are the (j,j)-entry of matrix $\tilde{B}$ and M, respectively. A simple calculation gives

$$\tilde{b}_1 = \frac{1}{2}\left(\rho_1 b_1 - \frac{K'}{K} + \frac{\rho_1'}{\rho_1}\right), \qquad \tilde{b}_2 = \frac{1}{2}\left(\rho_2 b_2 - \frac{EI'}{EI} + \frac{\rho_2'}{\rho_2}\right),$$
$$\tilde{b}_3 = \frac{1}{2}\left(-\rho_1 b_1 - \frac{K'}{K} + \frac{\rho_1'}{\rho_1}\right), \quad \tilde{b}_4 = \frac{1}{2}\left(-\rho_2 b_2 - \frac{EI'}{EI} + \frac{\rho_2'}{\rho_2}\right).$$

From [2], we know that $\xi_{kj}(x,\lambda)$ and $\eta_{kj}(x,\lambda)$ are analytic for $\lambda \in G^-$ and are bounded uniformly for $\lambda \in G^-$ and $x \in [0,\ell]$.

Denote $v_p(x) = [p_1(x), p_2(x), p_3(x), p_4(x)]^\tau$ and

$$v_c(x) = T_c(x)\Theta = [r_1(x), r_2(x), r_3(x), r_4(x)]^\tau.$$

It is easily seen that

$$p_k(x) = e^{\mu_k(x)}q_k(x) + \lambda^{-1}\sum_{j=1}^{4}\xi_{kj}(x,\lambda)e^{\mu_j(x)}q_j(x), \quad 1 \le k \le 4, \tag{19}$$

$$q_j(x) = \int_0^x e^{-\mu_j(s)}\left[h_j(s) + \lambda^{-1}\sum_{n=1}^{4}\eta_{jn}(s,\lambda)h_n(s)\right]ds, \quad 1 \le j \le 4, \tag{20}$$

$$r_k(x) = e^{\mu_k(x)}\theta_k + \lambda^{-1}\sum_{j=1}^{4}\xi_{kj}(x,\lambda)e^{\mu_j(s)}\theta_j, \quad 1 \le k \le 4, \tag{21}$$

where $h_1 = h_3 = -\rho_1^2(w_1 + (b_1 w_1 + z_1)/\lambda)$ and $h_2 = h_4 = -\rho_2^2(\varphi_1 + (b_2\varphi_1 + \psi_1)/\lambda)$.

3. Asymptotic distribution of eigenvalues of the system

In this section, we assume that conditions (8) and (9) are satisfied. The well-known Rouche's theorem will be used to investigate the asymptotic behaviour of the eigenvalues of the system (5). The method used is similar to that given in [18].

Substituting $v(x) = T_c(x)\Theta + v_p(x)$ into the boundary conditions of (13), we get

$$\begin{bmatrix}\tilde{B}_1 T_c(0)\\ \tilde{B}_2 T_c(\ell)\end{bmatrix}\Theta = -\begin{bmatrix}\tilde{B}_1 v_p(0)\\ \tilde{B}_2 v_p(\ell)\end{bmatrix} \triangleq \tilde{\Theta}. \tag{22}$$

Multiplying (22) by

$$N=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1/\lambda&0\\0&0&0&1\end{bmatrix},$$

we have

$$(\hat{E}+\lambda^{-1}\hat{E}_1)\Theta=\hat{\Theta}, \tag{23}$$

where

$$\hat{E}=\begin{bmatrix}\tilde{B}_1\\ \hat{B}_2(\exp(\mu_j(\ell))\delta_{kj})_{4\times4}\end{bmatrix},\quad \hat{E}_1=\begin{bmatrix}\tilde{B}_1(\xi_{kj}(0,\lambda))_{4\times4}\\ \hat{B}_2(\exp(\mu_j(\ell))\xi_{kj}(\ell,\lambda))_{4\times4}\end{bmatrix}, \tag{24}$$

$$\hat{B}_2=\begin{bmatrix}-1&1/\lambda\rho_2(\ell)&-1&-1/\lambda\rho_2(\ell)\\0&1&0&1\end{bmatrix},$$

$$\hat{\Theta}=[0,0,(p_1(\ell)+p_3(\ell))-(p_2(\ell)-p_4(\ell))/(\lambda\rho_2(\ell)),\ -(p_2(\ell)+p_4(\ell))]^{\tau}.$$

It is easy to see that $\lambda\in G^-$ is an eigenvalue of $\mathscr{A}_1$ if and only if

$$\det\left(\hat{E}+\lambda^{-1}\hat{E}_1\right)=0, \tag{25}$$

where $\hat{E}$, $\hat{E}_1$ are given by (24). In terms of the properties of the determinants, we have

$$\det\left(\hat{E}+\lambda^{-1}\hat{E}_1\right)=\det\hat{E}+\lambda^{-1}E_0(\lambda). \tag{26}$$

Since $\xi_{kj}(x,\lambda)$, $1\le k,\ j\le 4$, are analytic for $\lambda\in G^-$ and are bounded uniformly for $\lambda\in G^-$ and $x\in[0,\ell]$, $E_0(\lambda)$ is analytic and uniformly bounded for $\lambda\in G^-$.

By a simple calculation, we get

$$\det\hat{E}=-\frac{[e^{\mu_1(\ell)}+e^{\mu_3(\ell)}][e^{\mu_2(\ell)}+e^{\mu_4(\ell)}]}{\rho_1(0)\rho_2(0)}. \tag{27}$$

Thus we see that $\det\hat{E}=0$ is equivalent to

$$e^{\mu_1(\ell)}+e^{\mu_3(\ell)}=0\quad\text{or}\quad e^{\mu_2(\ell)}+e^{\mu_4(\ell)}=0. \tag{28}$$

The roots of (28) are

$$\lambda_{jn}=-\frac{\int_0^\ell\rho_jb_j\,ds-\boldsymbol{i}(2n+1)\pi}{2\int_0^\ell\rho_j\,ds},\qquad j=1,2,\ n\in\mathbb{Z},$$

where $\mathbb{Z}$ is the integer set. Now, for large positive integer n, define two rectangles as follows:

$$\Lambda_{jn}=\left\{\lambda\in G^-\ \middle|\ |\operatorname{Re}(\lambda-\lambda_{jn})|=1/\sqrt{n},\ |\operatorname{Im}(\lambda-\lambda_{jn})|=1/\sqrt{n}\right\},\quad j=1,2. \tag{29}$$

To apply Rouche's theorem, we need to estimate $\det \hat{E}$ on Λ_{jn} for $j = 1, 2$. In the following, we assume that

$$\frac{\int_0^\ell \rho_2 b_2\, ds}{\int_0^\ell \rho_2\, ds} \neq \frac{\int_0^\ell \rho_1 b_1\, ds}{\int_0^\ell \rho_1\, ds}. \tag{30}$$

For simplicity, we consider only the case in which $\lambda \in \Lambda_{1n}$, $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. Other cases can be established similarly.

Set $\Lambda_{1n}^0 = \{\lambda \in G^- \mid \operatorname{Re}\lambda = \operatorname{Re}\lambda_{1n} \pm 1/\sqrt{n},\ |\operatorname{Im}(\lambda - \lambda_{1n})| \leq 1/\sqrt{n}\}$. If $\lambda \in \Lambda_{1n}^0$, then, by a simple calculation, we obtain

$$\operatorname{Re}\mu_3(\ell) = \int_0^\ell \tilde{b}_3(s)\, ds + \frac{1}{2}\int_0^\ell \rho_1(s) b_1(s)\, ds \mp \frac{1}{\sqrt{n}}\int_0^\ell \rho_1(s)\, ds, \tag{31}$$

$$\operatorname{Re}\mu_4(\ell) = \int_0^\ell \tilde{b}_4(s)\, ds + \frac{1}{2}\int_0^\ell \rho_2\, ds \frac{\int_0^\ell \rho_1 b_1\, ds}{\int_0^\ell \rho_1\, ds} \mp \frac{1}{\sqrt{n}}\int_0^\ell \rho_2(s)\, ds, \tag{32}$$

$$\operatorname{Re}(\mu_1(\ell) - \mu_3(\ell)) = \pm\frac{2}{\sqrt{n}}\int_0^\ell \rho_1\, ds, \tag{33}$$

$$\operatorname{Re}(\mu_2(\ell) - \mu_4(\ell)) = \int_0^\ell \rho_2 b_2\, ds - \int_0^\ell \rho_2\, ds \frac{\int_0^\ell \rho_1 b_1\, ds}{\int_0^\ell \rho_1\, ds} \pm \frac{2}{\sqrt{n}}\int_0^\ell \rho_2\, ds. \tag{34}$$

It follows from (31) and (32) that $\operatorname{Re}\mu_3(\ell)$ and $\operatorname{Re}\mu_4(\ell)$ are uniformly bounded with respect to n. Thus there exists a $\delta_1 > 0$ such that

$$\left|\exp(\mu_3(\ell))\exp(\mu_4(\ell))\right| \geq \delta_1. \tag{35}$$

Using the elementary inequality

$$\left|e^{\sigma_1 + i\tau} + 1\right| \geq |e^{\sigma_1} - 1| \geq \begin{cases} \sigma_1, & \sigma_1 \geq 0; \\ |\sigma_1| - \sigma_1^2/2, & -2 < \sigma_1 < 0, \end{cases} \tag{36}$$

with $\sigma_1 = \operatorname{Re}(\mu_1(\ell) - \mu_3(\ell))$, we obtain from (33) that there exist $n_1, \delta_2 > 0$ such that

$$\left|\exp(\mu_1(\ell) - \mu_3(\ell)) + 1\right| > \delta_2/\sqrt{n} \tag{37}$$

provided that $n \geq n_1$.

Since $\left|\exp(\mu_2(\ell) - \mu_4(\ell)) + 1\right| \geq \left|\exp(\operatorname{Re}(\mu_2(\ell) - \mu_4(\ell)) - 1\right|$ and

$$\operatorname{Re}(\mu_2(\ell) - \mu_4(\ell)) \to \int_0^\ell \rho_2 b_2\, ds - \int_0^\ell \rho_2\, ds \int_0^\ell \rho_1 b_1 ds \left(\int_0^\ell \rho_1\, ds\right)^{-1} \neq 0,$$

as $n \to \infty$ there exist $n_2, \delta_3 > 0$ such that

$$|\exp(\mu_2(\ell) - \mu_4(\ell)) + 1| > \delta_3, \quad \text{for } n > n_2. \tag{38}$$

Therefore, substituting (35), (37) and (38) into (27), we have

$$|\det \hat{E}| > \frac{\delta_1 \delta_2 \delta_3}{\rho_1(0)\rho_2(0)\sqrt{n}}, \quad \forall n > n_3 = \max\{n_0, n_1, n_2\}. \tag{39}$$

Furthermore, it is easy to see that there exist $n_4, \delta_4 > 0$ such that

$$\left|\lambda^{-1} E_0(\lambda)\right| < \delta_4/n, \quad \forall n > n_4. \tag{40}$$

Combining (39) and (40), we get

$$\left|\lambda^{-1} E_0(\lambda)\right| < |\det \hat{E}| \quad \forall \lambda \in \Lambda_{1n}^0, \; n > \max\{n_3, n_4\}. \tag{41}$$

On the other hand, if

$$\lambda \in \Lambda_{1n}^1 = \left\{\lambda \in G^- \mid \operatorname{Im}\lambda = \operatorname{Im}\lambda_{1n} \pm 1/\sqrt{n}, |\operatorname{Re}(\lambda - \lambda_{1n})| \leq 1/\sqrt{n}\right\},$$

then

$$\begin{aligned}
\operatorname{Re}\mu_3(\ell) &= \int_0^\ell \tilde{b}_3(s)\,ds + \frac{1}{2}\int_0^\ell \rho_1(s) b_1(s)\,ds + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right), \\
\operatorname{Re}\mu_4(\ell) &= \int_0^\ell \tilde{b}_4(s)\,ds + \frac{1}{2}\int_0^\ell \rho_2\,ds \frac{\int_0^\ell \rho_1 b_1\,ds}{\int_0^\ell \rho_1\,ds} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right), \\
\operatorname{Im}(\mu_1(\ell) - \mu_3(\ell)) &= \pm\frac{2}{\sqrt{n}}\int_0^\ell \rho_1\,ds + (2n+1)\pi, \\
\operatorname{Re}(\mu_2(\ell) - \mu_4(\ell)) &= \int_0^\ell \rho_2 b_2\,ds - \int_0^\ell \rho_2\,ds \frac{\int_0^\ell \rho_1 b_1\,ds}{\int_0^\ell \rho_1\,ds} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

It is easy to see that there exist $n_5, \delta_5 > 0$ such that

$$|\exp(\mu_1(\ell) - \mu_3(\ell)) + 1| \geq \left|\sin\left(\frac{2}{\sqrt{n}}\int_0^\ell \rho_1 ds\right)\right| \geq \frac{\delta_5}{\sqrt{n}}, \quad \forall n \geq n_5,$$

where we have used the elementary inequality $|\exp(\sigma + \boldsymbol{i}\tau) + 1| \geq |\sin\tau|$ with $\tau = \operatorname{Im}(\mu_1(\ell) - \mu_3(\ell))$. Therefore, similar to the case of $\lambda \in \Lambda_{1n}^0$, we know that there exists an $n_6 > 0$ such that

$$\left|\lambda^{-1} E_0(\lambda)\right| < |\det \hat{E}|, \quad \forall \lambda \in \Lambda_{1n}^1, \; n > n_6. \tag{42}$$

THEOREM 2. *Assume that* λ *is an eigenvalue of the system* (5). *If* $|\lambda|$ *is sufficiently large and condition* (30) *holds, then there exists some* n *such that* $|\lambda - \lambda_{jn}| < 1/\sqrt{n}$, $j = 1$ *or* 2, *where* n *is dependent only on* λ. *Furthermore,* λ *satisfies*

$$\lambda = -\frac{\int_0^\ell \rho_j b_j \, ds - \boldsymbol{i}(2n+1)\pi}{2\int_0^\ell \rho_j \, ds} + \mathscr{O}\left(\frac{1}{\sqrt{n}}\right), \quad j = 1, 2,$$

for large n.

PROOF. From (41) and (42), we obtain

$$\left|\det(\hat{E} + \lambda^{-1}\hat{E}_1) - \det \hat{E}\right| < \left|\det \hat{E}\right|, \quad \forall \lambda \in \Lambda_{jn} = \Lambda_{jn}^0 \cup \Lambda_{jn}^1, \ j = 1, 2,$$

provided that n is large enough. Since $\det(\hat{E} + \lambda^{-1}\hat{E}_1)$ and $\det \hat{E}$ are analytic for $\lambda \in \Lambda_{jn}$, $n \geq n_0$, $j = 1, 2$, it follows from Rouche's theorem that $\det(\hat{E} + \lambda^{-1}\hat{E}_1)$ and $\det \hat{E}$ have the same number of zeros inside the rectangle Λ_{jn}. On the other hand, it is easy to prove that there are no zeros of $\det(\hat{E} + \lambda^{-1}\hat{E}_1)$ outside $\Lambda_{1n} \cup \Lambda_{2n}$ when n is large enough. Thus the desired result is obtained.

REMARK 1. If $\int_0^\ell \rho_1 b_1 \, ds \big/ \int_0^\ell \rho_1 \, ds = \int_0^\ell \rho_2 b_2 \, ds \big/ \int_0^\ell \rho_2 \, ds$ and $\int_0^\ell \rho_1 \, ds \big/ \int_0^\ell \rho_2 \, ds$ is an irrational number, then by a well-known result of number theory (see [16, Theorem 7.9]),

$$\inf_{n \neq m} |\lambda_{1n} - \lambda_{2m}| = \inf_{n \neq m} \left| \frac{\boldsymbol{i}(2n+1)\pi}{2\int_0^\ell \rho_1 \, ds} - \frac{\boldsymbol{i}(2m+1)\pi}{2\int_0^\ell \rho_2 \, ds} \right| = 0,$$

which means that there is no gap between the zeros of $\det \hat{E}$. In this case, we cannot use Rouche's theorem to obtain the explicit asymptotic expressions for the eigenvalues as for the case given in Theorem 2.

4. Energy exponential decay rate of the closed loop system

In the following discussion, we denote by $s(\mathscr{A})$ and $\omega(\mathscr{A})$ the spectrum bound of $\mathscr{A}$ and the growth bound of the semigroup $T(t)$ generated by $\mathscr{A}$, respectively.

In [8], it is shown that for a uniformly bounded C_0-semigroup $T(t)$ on a Hilbert space $\mathscr{H}$, the spectrum-determined growth assumption holds, that is, $s(\mathscr{A}) = \omega(\mathscr{A})$ and $T(t)$ is exponentially stable if and only if

$$\{\boldsymbol{i}\omega \mid \omega \in \mathbb{R}\} \subset \rho(\mathscr{A}), \tag{43}$$

and for every $\sigma_0 \in (s(\mathscr{A}), 0]$,

$$K_1 \triangleq \sup\left\{\|(\lambda - \mathscr{A})^{-1}\| \mid \operatorname{Re}\lambda \in [\sigma_0, 0]\right\} < \infty, \tag{44}$$

where $\mathscr{A}$ is the infinitesimal generator of $T(t)$. To verify the spectrum-determined growth assumption and the exponential stability of the semigroup $T(t)$ generated by $\mathscr{A}_1$, it is sufficient to show that (43) and (44) hold, because the semigroup $T(t)$ generated by $\mathscr{A}_1$ is uniformly bounded in terms of Theorem 1.

LEMMA 2. *The imaginary axis is a subset of the resolvent set of $\mathscr{A}_1$, $\rho(\mathscr{A}_1)$, that is, $\{\boldsymbol{i}\omega \mid \omega \in \mathbb{R}\} \subset \rho(\mathscr{A}_1)$.*

PROOF. Otherwise there exists an $\omega_0 \neq 0$ (because $0 \in \rho(\mathscr{A}_1)$) such that $\boldsymbol{i}\omega_0 \notin \rho(\mathscr{A}_1)$. Since $\mathscr{A}$ has a compact resolvent, we have $\boldsymbol{i}\omega_0 \in \sigma_p(\mathscr{A}_1)$. Thus $\exists Y_0 = [w_0, z_0, \varphi_0, \psi_0]^\tau \in \mathscr{D}(\mathscr{A}_1)$ with $Y_0 \neq 0$, satisfying

$$\operatorname{Re}(\mathscr{A}Y_0, Y_0) + \operatorname{Re}(\mathscr{B}Y_0, Y_0) = \operatorname{Re}(\boldsymbol{i}\omega_0)\|Y_0\|^2_{\mathscr{H}} = 0.$$

Hence $\operatorname{Re}(\mathscr{B}Y_0, Y_0)_{\mathscr{H}} = 0$. It follows from (7) that $\mathscr{B}Y_0 = 0$, which, in turn, implies that $\mathscr{A}Y_0 = \boldsymbol{i}\omega_0 Y_0$, that is,

$$\begin{cases} z_0 = \boldsymbol{i}\omega_0 w_0, \\ -(K(\varphi_0 - w_0'))' = \boldsymbol{i}\omega_0 \rho z_0, \\ \psi_0 = \boldsymbol{i}\omega_0 \varphi_0, \\ (EI\varphi_0')' - K(\varphi_0 - w_0') = \boldsymbol{i}\omega_0 I_\rho \psi_0, \\ \varphi_0(\ell) = w_0'(\ell), \quad \varphi_0'(\ell) = 0, \end{cases} \tag{45}$$

and $z_0(x) = \psi_0(x) = 0\ \forall x \in [c, d]$. From the first and third equations of (45), it follows that $w_0(x) = \varphi_0(x) = 0\ \forall x \in [c, d]$. Thus (45) has only the zero solution, which contradicts $Y_0 \neq 0$. The proof is complete.

Next we proceed to show that (44) holds for $\mathscr{A}_1$ with the help of the asymptotic solutions of the resolvent equation obtained in Section 2.

Let c_k be a positive constant for $k = 1, 2, \ldots$ and define $H = (L^2[0, \ell])^4$. For every $\sigma_0 \in (s(\mathscr{A}_1), 0]$, let $\lambda = \sigma + \boldsymbol{i}\omega$ for a real number ω and $\sigma \in [\sigma_0, 0]$. It follows from (10) that for $Y_1 = [w_1, z_1, \varphi_1, \psi_1]^\tau \in \mathscr{H}$,

$$\begin{aligned} \left\|(\lambda\mathscr{I} - \mathscr{A}_1)^{-1}Y_1\right\|^2_{\mathscr{H}} &= \int_0^\ell K|\varphi - w'|^2\,dx + \int_0^\ell EI|\varphi'|^2\,dx \\ &\quad + \int_0^\ell \rho|\lambda w - w_1|^2\,dx + \int_0^\ell I_\rho|\lambda\varphi - \varphi_1|^2\,dx \\ &\leq c_1\left(\|Y_1\|^2_{\mathscr{H}} + \|\lambda u\|^2_H\right), \end{aligned} \tag{46}$$

where u is determined by (12) with $\lambda = \sigma + \boldsymbol{i}\omega$.

On the other hand, by virtue of the triangle inequality, we have

$$\|\lambda u\|_H^2 = \|\lambda Q v\|_H^2 \le c_2 \left(\sum_{k=1}^{4} \left(\|\lambda p_k\|_{L^2[0,\ell]} + \|\lambda r_k\|_{L^2[0,\ell]} \right) \right)^2, \quad |\lambda| > N. \tag{47}$$

If the following estimates for large $|\lambda|$ are satisfied,

$$\|\lambda p_k\|_{L^2[0,\ell]} \le c_3 \|Y_1\|_{\mathscr{H}}, \tag{48}$$

$$\|\lambda r_k\|_{L^2[0,\ell]} \le c_3 \|Y_1\|_{\mathscr{H}}, \tag{49}$$

then by taking into account the continuity of $\|(\lambda - \mathscr{A}_1)^{-1}\|_{\mathscr{H}}$ with respect to $\lambda \in \rho(\mathscr{A}_1)$, it follows from (46) and (47) that (44) holds.

Note that there exist $M_0, M_1 > 0$ such that for $\sigma \in [\sigma_0, 0]$ and $1 \le j \le 4$,

$$\left| e^{\pm \mu_j(x)} \right| = \left| e^{\pm \int_0^x \tilde{b}_j(s)\, ds \pm \sigma \int_0^x m_j(s)\, ds} \right| \le M_0, \quad \forall x \in [0, \ell], \tag{50}$$

$$\sup_{1 \le k, j \le 4} \left\{ |\xi_{kj}(x, \lambda)|, |\eta_{kj}(x, \lambda)| \right\} \le M_1, \quad \forall x \in [0, \ell],\ \lambda \in G^-. \tag{51}$$

First, we verify (48) for $|\lambda| > N$. For this, we prove (48) only for $k = 1$ and the proofs for other cases are similar. Using (19) and the triangle inequality, we obtain $\left(\int_0^\ell |\lambda p_1(x)|^2\, dx \right)^{1/2} \le J_1 + J_2$, where

$$J_1 = \left(\int_0^\ell \left| \lambda e^{\mu_1(x)} q_1(x) \right|^2 dx \right)^{1/2}, \quad J_2 = \sum_{j=1}^{4} \left(\int_0^\ell \left| \xi_{1j}(x, \lambda) e^{\mu_j(x)} q_j(x) \right|^2 dx \right)^{1/2}.$$

Similarly, it follows from (20) that $J_1 \le M_0(J_{11} + J_{12})$, where

$$J_{11} = \left(\int_0^\ell \left| \int_0^x \lambda e^{-\mu_1(s)} h_1(s)\, ds \right|^2 dx \right)^{1/2},$$

$$J_{12} = \sum_{n=1}^{4} \left(\int_0^\ell \left| \int_0^x \eta_{1n}(s, \lambda) h_n(s) e^{-\mu_1(s)}\, ds \right|^2 dx \right)^{1/2}.$$

Then, by the definition of h_n and the triangle inequality, we have

$$\begin{aligned} J_{11} \le{}& \left(\int_0^\ell \left| \int_0^x \lambda e^{-\mu_1(s)} \rho_1^2(s) w_1(s)\, ds \right|^2 dx \right)^{1/2} \\ &+ \left(\int_0^\ell \left| \int_0^x e^{-\mu_1(s)} \rho_1^2 (b_1 w_1 + z_1)\, ds \right|^2 dx \right)^{1/2}. \end{aligned} \tag{52}$$

Integrating by parts yields

$$\int_0^x \lambda e^{-\mu_1(s)} \rho_1^2(s) w_1(s)\, ds = \int_0^x G_\lambda(x,s)(\rho_1^2 w_1)'(s)\, ds, \tag{53}$$

where

$$\begin{aligned} G_\lambda(x,s) &= \int_s^x \lambda e^{-\mu_1(t)}\, dt = \int_s^x -\frac{1}{\rho_1}\frac{d}{dt}\left(e^{-\lambda\int_0^t \rho_1(\sigma)\,d\sigma}\right) e^{-\int_0^t \tilde{b}_1(\sigma)\,d\sigma}\, dt \\ &= -\left.\frac{e^{-\mu_1(t)}}{\rho_1(t)}\right|_s^x - \int_s^x \frac{\tilde{b}_1(t)\rho_1(t)+\rho_1'(t)}{\rho_1^2(t)} e^{-\mu_1(t)}\, dt, \quad 0 \le s \le x \le \ell. \end{aligned}$$

Then it follows from (8) and (50) that $G_\lambda(x,s)$ is uniformly bounded, that is, $\exists M_2 > 0$, such that $|G_\lambda(x,s)| \le M_2$ provided that $0 \le s \le x \le \ell$ and $\lambda = \sigma + \boldsymbol{i}\omega$ with $\sigma \in [\sigma_0, 0]$. Hence, by Schwarz's inequality, we get

$$\left| \int_0^x G_\lambda(x,s)(\rho_1^2 w_1)'(s)\, ds \right| \le c_4(\|w_1'\|_{L^2[0,\ell]} + \|w_1\|_{L^2[0,\ell]}). \tag{54}$$

Substituting (54) into (52), and again using Schwarz's inequality, we obtain

$$J_{11} \le c_5 \|Y_1\|_{\mathscr{H}}. \tag{55}$$

Similarly, we obtain

$$J_{12} \le M_0 M_1 \ell \sum_{n=1}^{4} \left(\int_0^\ell |h_n(s)|^2\, ds \right)^{1/2} \le c_6 \|Y_1\|_{\mathscr{H}}. \tag{56}$$

Thus it follows from (20), (50), (51) and the triangle inequality that

$$J_2 \le c_7 \|Y_1\|_{\mathscr{H}}. \tag{57}$$

Combining (55)–(57), we conclude that (48) holds.

Now we turn to verifying (49) for large $|\lambda|$. By the proof of Theorem 2, we get

$$\sigma_0 > s(\mathscr{A}_1) \ge \max\left\{ -\frac{\int_0^\ell \rho_j b_j\, ds}{2\int_0^\ell \rho_j\, ds} \;\middle|\; j = 1,2 \right\}.$$

Thus using (36) with $\sigma_1 = \mathrm{Re}(\mu_1(\ell) - \mu_3(\ell))$, we obtain

$$\begin{aligned} \left|e^{\mu_1(\ell)} + e^{\mu_3(\ell)}\right| &= \left|e^{\mu_3(\ell)}\right| \left|e^{\mu_1(\ell)-\mu_3(\ell)} + 1\right| \\ &\ge e^{\int_0^\ell \tilde{b}_3(s)\, ds} \left(\int_0^\ell \rho_1 b_1\, ds + 2\sigma_0 \int_0^\ell \rho_1\, ds \right) > 0. \end{aligned} \tag{58}$$

By a similar argument, we have

$$\left|e^{\mu_2(\ell)} + e^{\mu_4(\ell)}\right| \ge e^{\int_0^\ell \tilde{b}_4(s)\, ds} \left(\int_0^\ell \rho_2 b_2\, ds + 2\sigma_0 \int_0^\ell \rho_2\, ds \right) > 0. \tag{59}$$

Thus it follows from (27), (58) and (59) that $|\det \hat{E}| > c_8 > 0$, and hence $\hat{E}^{-1}$ exists. So (23) can be written as $\left(I_4 + \lambda^{-1}\hat{E}^{-1}\hat{E}_1\right)\Theta = \hat{E}^{-1}\hat{\Theta}$. Since the entries of $\hat{E}$ and $\hat{E}_1$ are uniformly bounded with respect to $\lambda \in G^-$ and $|\det \hat{E}| > c_8$, the entries of $\hat{E}^{-1}$ and $\hat{E}^{-1}\hat{E}_1$ are also uniformly bounded with respect to $\lambda \in G^-$. Hence $\left(I_4 + \lambda^{-1}\hat{E}^{-1}\hat{E}_1\right)^{-1}$ exists provided that $|\lambda|$ is sufficiently large, and

$$\Theta = \left(I_4 + \lambda^{-1}\hat{E}^{-1}\hat{E}_1\right)^{-1}\hat{E}^{-1}\hat{\Theta}. \tag{60}$$

Thus it follows from (60) that there exists an $N_1 > 0$, such that

$$\sum_{k=1}^{4} |\lambda\theta_k| \le c_9 \sum_{k=1}^{4} |\lambda p_k|, \quad |\lambda| > N_1. \tag{61}$$

Using (19), (50) and (51), we obtain

$$|\lambda p_k(\ell)| \le M_0|\lambda q_k(\ell)| + M_0 M_1 \sum_{j=1}^{4} |q_j(\ell)|, \qquad 1 \le k \le 4. \tag{62}$$

Using (53) and (54) with $x = \ell$ and Schwarz's inequality, we have

$$|\lambda q_k(\ell)| \le c_{11}\|Y_1\|_{\mathscr{H}}, \qquad 1 \le k \le 4. \tag{63}$$

Substituting (63) into (62), we get

$$|\lambda p_k(\ell)| \le c_{12}\|Y_1\|_{\mathscr{H}}, \quad \text{for } |\lambda| \ge N_1,\ 1 \le k \le 4. \tag{64}$$

Therefore it follows from (21), (50) and (51) that

$$\left(\int_0^{\ell} |\lambda r_k(x)|^2\, dx\right)^{1/2} \le M_0\sqrt{\ell}|\lambda\theta_k| + M_0 M_1\sqrt{\ell}\sum_{j=1}^{4}|\theta_j|. \tag{65}$$

Finally, by substituting (64) into (61) and then into (65), we obtain (49). Thus we prove the main result of this paper.

THEOREM 3. *With the assumptions (4), (8) and (9), the spectrum-determined growth assumption for the closed loop system (5) holds, that is,* $s(\mathscr{A}_1) = \omega(\mathscr{A}_1)$. *Furthermore, the* C_0*-semigroup* $T(t)$ *generated by* $\mathscr{A}_1$ *is exponentially stable.*

REMARK 2. By using the frequency domain multiplier method, we can prove that the closed loop system is exponentially stable without the assumption of different wave speeds, which will be reported in another article.

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References

[1] H. T. Banks, R. C, Smith and Yun Wang, "The modeling of piezoceramic patch interactions with shells, plates, and beams", *Quart. Appl. Math.* **53** (1995) 353–381.

[2] G. D. Birkhoff and R. E. Langer, "The boundary problems and developments associated with a system of ordinary linear differential equations of the first order", *Proc. Amer. Acad. Arts Sci.* **2** (1923) 51–128.

[3] G. Chen, S. G. Krantz, D. W. Ma, C. E. Wayne and H. H. West, "The Euler-Bernoulli beam equation with boundary energy dissipation", in *Operator Methods for Optimal Control Problems* (ed. S. J. Lee), Lecture Notes in Pure and Appl. Math. Series, (Marcel Dekker, New York, 1987) 67–96.

[4] G. Chen, M. C. Delfour, A. M. Krall and G. Payre, "Modeling, stabilization and control of serially connected beams", *SIAM J. Control Optim.* **25** (1987) 526–546.

[5] G. Chen, S. A. Fulling, F. J. Narcowich and S. Sun, "Exponential decay of energy of evolution equations with locally distributed damping", *SIAM J. Appl. Math.* **51** (1991) 266–301.

[6] W. Desch, K. B. Hannsgen, Y. Renardy and R. L. Wheeler, "Boundary stabilization of an Euler-Bernoulli beam with viscoelastic damping", in *Proc. IEEE Conf. on Decision and Control*, (Los Angeles, CA, 1987) 1792–1795.

[7] L. F. Ho, "Exact controllability of the one-dimensional wave equation with locally distributed control", *SIAM J. Control Optim.* **28** (1990) 733–748.

[8] F. L. Huang, "Characteristic conditions for exponential stability of linear dynamical system in Hilbert spaces", *Ann. Diff. Eqs.* **1** (1985) 43–53.

[9] J. U. Kim and Y. Renardy, "Boundary control of the Timoshenko beam", *SIAM J. Control Optim.* **25** (1987) 1417–1429.

[10] J. Lagnese, "Control of wave processes with distributed controls supported on a subregion", *SIAM J. Control Optim.* **21** (1983) 68–85.

[11] K. S. Liu, "Locally distributed control and damping for the conservative systems", *SIAM J. Control Optim.* **35** (1997) 1574–1590.

[12] K. S. Liu, F. L. Huang and G. Chen, "Exponential stability analysis of a long chain of coupled vibrating strings with dissipative linkage", *SIAM J. Appl. Math.* **49** (1989) 1694–1707.

[13] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer-Verlag, New York, 1983).

[14] D. L. Russell, "Controllability and stabilizability theory for linear partial differential equations, recent progress and open questions", *SIAM Rev.* **20** (1978) 639–740.

[15] D. L. Russell, "Mathematical models for the elastic beam and their control-theoretic implications", in *Semigroups, Theory and Applications* (eds. H. Brezis, H. G. Crandall and F. Kapell), Volume 2, (Longman, Essex, 1986) 177–217.

[16] H. M. Stark, *An Introduction to Number Theory* (The MIT Press, Cambridge, Massachusetts, 1979).

[17] S. Timoshenko, *Vibration Problems in Engineering* (Van Nostand, New York, 1955).

[18] H. K. Wang and G. Chen, "Asymptotic locations of eigenfrequencies of Euler-Bernoulli beam with nonhomogeneous structural and viscous damping coefficients", *SIAM J. Control Optim.* **29** (1991) 347–367.