NEWTON’S DIAGRAM METHOD FOR NONLINEAR EQUATIONS WITH SEVERAL SMALL PARAMETERS

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Abstract

In this article, we generalise Newton’s diagram method for finding small solutions \( \xi(\lambda) \) of equations \( f(\xi, \lambda) = 0 \) (\( f(0, 0) = 0 \)) with \( f \) analytic (see [4, 6, 1, 2]) to the case of a multi-dimensional function \( f \), unknown variable \( \xi \) and small parameter \( \lambda \). This method was briefly described in [1]. The method has many different applications and allows one to solve some inflexible problems. In particular, the method can be used in very difficult bifurcation problems, for example, for systems with small imperfections.

1. Introduction

In this paper, we propose a new method for finding all the small solutions \(^2\) (SS) of equations \( f(\xi, \lambda) = 0 \) (\( f(0, 0) = 0 \)) with \( f \) analytic, where the small parameter \( \lambda \) is multi-dimensional: \( \lambda = (\lambda_1, \ldots, \lambda_n) \). This method has many different applications and allows one to solve a number of inflexible problems. In particular, it can be very useful in problems related to dynamic systems (bifurcations of equilibrium states, boundedness and stability of limit cycles), deformed systems, etc., which can be reduced to the above defined problem. Some examples of typical problems of static systems are presented in [5] and in [6, pp. 454–469].

In this article, the method is described in three sections. In Section 2 the method is explicated for two small parameters \((n = 2)\). In Section 3 we consider the critical situation when the method does not give the SS but, nevertheless, the problem can be solved by means of some additional considerations. In Section 4 the method is generalised for \( n (n > 2) \) small parameters.

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\(^2\)The continuous solution \( \xi(\lambda) \) is defined to be an SS if \( \xi(\lambda) \to 0 \) as \( \lambda \to 0 \).
2. Case of two parameters \((n = 2)\)

First, we describe this method for an equation with two small scalar parameters: \(\lambda_1\) and \(\lambda_2\), that is, when \(\lambda\) is two-dimensional:

\[
\Gamma(\xi, \lambda_1, \lambda_2) \equiv \sum_{m+n+k=1}^{\infty} \varphi_{mnk}(\xi)\lambda_1^m\lambda_2^k = 0, \tag{1}
\]

where \(\xi = (\xi_1, \ldots, \xi_s), \varphi_{mnk}(\xi) = \{\varphi^{(1)}_{mnk}(\xi), \ldots, \varphi^{(s)}_{mnk}(\xi)\}, \varphi^{(i)}_{mnk}(\xi)\) is a scalar homogeneous polynomial relative to \(\xi_1, \ldots, \xi_s, \) of degree \(m\) with real coefficients or a zero polynomial. Let us assume that \(\varphi_{m00}(\xi) = 0\) at \(m < m_0\) and \(\varphi_{mm0}(\xi) \neq 0\).

It is also assumed that the series \(\Gamma\) converges in some neighbourhood of zero. We are studying the problem about SS \(\xi(\lambda_1, \lambda_2)\) of (1) for \((\lambda_1, \lambda_2)\) near the straight line \(\lambda_1 = 0\) but away from the line \(\lambda_2 = 0\). The indicated neighbourhood \(T\) must satisfy the condition that \((0, 0) \in \overline{T}\) and \(|\lambda_1/\lambda_2|\) is bounded in \(T; s \in \mathbb{N}\). The minimal value of \(s\) is found from the method.

Let us denote \(OMNK\) to be a Cartesian coordinate system in \(\mathbb{R}^3; \pi_1\) and \(\pi_2\) to be operators of the orthogonal projection of \(\mathbb{R}^3\) on the coordinate planes \(OMN\) and \(OMK\) respectively; \(Q = \{(m, n, k) : \varphi_{mnk}(\xi) \neq 0\}\) and \(Q_0 = \{(m, n, k) \in Q : m \leq m_0\}\) to be integer lattices; \(R_1 = \pi_1(Q_0)\) and \(R_2 = \pi_2(Q_0)\).

First of all, we construct Newton’s diagram (denoted by \(D_1\)) for \(R_1\) on the plane \(OMN\).

We denote its links and slopes as \(L^1, \ldots, L^x\) and \(p^{(1)}, \ldots, p^{(x)}\) respectively. Moreover, it is assumed that the links are numbered in order of the increase of their slopes (the count is made from right to left) so that \(0 \leq p^{(1)} < \cdots < p^{(x)}\). Further, let us construct Newton’s diagram on the plane \(OMK\) for the lattice \(\pi_2^{-1}(L^i) \cap Q_0, i \in \{1, \ldots, x\}\). Let us also denote its links and slopes as \(\{L'^i\}\) and \(\{p_{i,\theta}\}\) \((\theta = 1, \ldots, \alpha_i)\) respectively. If it has been done for all links \(L^i\) of \(D_1\), we get a broken line (a diagram) \(D_2\) on the plane \(OMK\).

Two cases can arise. The first one (the standard case), when \(D_2\) is the standard diagram for \(R_2\), that is, Newton’s diagram for \(R_2\) with a unique attenuation—the slopes corresponding to the adjacent links of \(D_1\) can coincide; \(p_{i,\theta} = p_{i+1,\theta}, i \in \{1, \ldots, x - 1\}\) and the second case (the general case) arises when \(D_2\) is not a standard diagram for \(R_2\).

Let us now show that by the conversion

\[
\lambda_1 = \mu_1\lambda_2^2, \quad \lambda_2 = \lambda_2, \tag{2}
\]

\(^1\)Newton’s diagram \(D_1\) for \(R_1\) is a convex broken line without vertical links. All tops of \(D_1\) belong to \(R_1\) and the links of \(D_1\) satisfy the minimal condition, that is, there are no points from \(R_1\) situated below the support lines of links. If the equation of the support line of a link is \(y = k(x - x_0)\), then the slope of this link is equal to \(-k\). All links of \(D_1\) (except maybe the right link) have positive slopes. The slope of the right link can be equal to 0. The Newton’s diagram method of construction is presented widely in the literature, see for example [6, pp. 10–15].
where \(s\) is a natural number, the general case reduces to the standard case. Indeed, making the substitution (2) in (1), we obtain
\[
\tilde{f}(\xi, \mu_1, \lambda_2) \equiv \sum_{m+n+k \geq 1} \varphi_{mnk}(\xi)\mu_1^{m}\lambda_2^{n+k} = 0, \tag{1'}
\]
where \(\varphi_{mnk}(\xi)\mu_1^{m}\lambda_2^{n+k}\) is the image of the term \(\varphi_{mnk}(\xi)\lambda_1^{n}\lambda_2^{k}\). Therefore (2) induces a linear conversion of the space of multi-indices \((m, n, k)\) with the matrix
\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & 1 \end{bmatrix}.
\]
Under that conversion the diagram \(D_1\) remains unchanged but the diagram \(D_2 = \{L^{i,\theta}\}\) transforms into the diagram \(\tilde{D}_2 = \{\tilde{L}^{i,\theta}\}\) and the slope \(\tilde{p}_{i,\theta}\) of the link \(\tilde{L}^{i,\theta}\) satisfies the relation
\[
\tilde{p}_{i,\theta} = p_{i,\theta} + sp(i).
\]
From (3) it follows that if we take \(s\) as the least non-negative entire number \(s_0\) satisfying the inequalities
\[
s_0 \geq \frac{p_{i,a_i} - p_{i+1,1}}{p^{\mu(i+1)} - p^{\mu(i)}}, \quad i = 1, \ldots, \alpha - 1, \tag{4}
\]
then all the links \(\tilde{D}_2\) will be arranged in non-decreasing order of their slopes (the count is made from right to left):
\[
\tilde{p}_{1,1} < \cdots < \tilde{p}_{1,a_i} \leq \tilde{p}_{2,1} < \cdots < \tilde{p}_{s-1,a_{s-1}} \leq \tilde{p}_{s,1} < \cdots < \tilde{p}_{s,a_s}.
\]
**NOTE 1.** If instead of \(s_0\) we take \(s_0 + 1\), then all links of \(\tilde{D}_2\) will be arranged in order of the increase of their slopes.

Let us suppose that \(P_i(m_i, k_i) (i = 1, \ldots, \alpha)\) is the set of all points from \(R_2\) situated under \(D_2\). For each point \(P_i\) let us put in accordance the point \((m_i, n'_i, k'_i)\) such that \((m_i, n'_i) \in D_i\) and \((m_i, k'_i) \in D_2\). By \(n_i\) we denote the least ordinate of the points of the set \(\pi_2^{-1}(P_i) \cap Q_0\), \(\Delta'_i = n'_i - n_i\) and \(\Delta_i = k'_i - k_i\). Let us use the conversion \(A\). Then the points \((m_i, n_i, k_i)\) and \((m_i, n'_i, k'_i)\) map to the points \((m_i, n_i, \tilde{k}_i)\) and \((m_i, n'_i, \tilde{k}'_i)\) accordingly, where \(\tilde{k}_i = k_i + sn_i\) and \(\tilde{k}'_i = k'_i + sn'_i\). Thus we obtain the relation
\[
\tilde{\Delta}'_i = \Delta'_i - s\Delta_i \quad (\tilde{\Delta}_i = \tilde{k}'_i - \tilde{k}_i). \tag{5}
\]
From (5) it follows that if \(s\) is replaced by the least natural number \(\tilde{s}\) satisfying the inequalities
\[
\tilde{s} \geq \Delta'_i/\Delta_i, \quad i = 1, \ldots, \alpha, \tag{6}
\]
then all points from \(\tilde{R}_2 = \pi_2(AQ_0)\) will not be situated under the diagram \(\tilde{D}_2\).
Note 2. If instead of \( s \) we take \( s + 1 \), then all points from \( R_2 \) lying below \( D_2 \) map to points situated above \( D_2 \). If \( s = \max(s_0, s) \), then by the conversion (2) the equation (1) is standardised, that is, it transforms into that for the standard case.

Let the equation (1) be standard. Let it be represented in the form

\[
\Gamma(\xi, \lambda_1, \lambda_2) \equiv \sum' \varphi_{\text{mak}}(\xi)\lambda_1^a \lambda_2^b + \sum'' \varphi_{\text{mak}}(\xi)\lambda_1^c \lambda_2^d = 0, \tag{1''}
\]

where \( \sum' \) denotes the sum of all non-zero terms of (1) such that \( (m, n) \in L' \) and \( (m, k) \in L'^{\alpha} \) (\( L' \) is any link of \( D_1 \) and \( L'^{\alpha} \) is one of the links of \( D_2 \) corresponding to \( L' \)) and \( \sum'' \) is the sum of the rest of the terms of (1). We denote also by \( r_1/s_1 \) the slope of \( L' \), by \( r_2/s_2 \) the slope of \( L'^{\alpha} \) and suppose that \( (r_1, s_1) = (r_2, s_2) = 1 \) ((a, b) denotes the greatest common divisor (GCD) for \( a \) and \( b \)). If \( r_1 = 0 \), then we assume \( s_1 = 1, i \in [1, 2] \).

Let us make the substitution

\[
\xi = \eta \mu_1^1 \mu_2^2, \quad \lambda_1 = \mu_1^1, \quad \lambda_2 = \mu_2^2. \tag{7}
\]

Then (1'') has been reduced to the form

\[
\sum' \varphi_{\text{mak}}(\eta)\mu_1^{mr_1} \mu_2^{mr_2} + \sum'' \varphi_{\text{mak}}(\eta)\mu_1^{mr_1} \mu_2^{mr_2} = 0. \tag{*}
\]

From the convexity of \( D_1 \) and the minimal condition of its links, it follows that \( mr_1 + ns_1 = l_1 \) for all \( (m, n) \in L' \cap R_1 \), where \( l_1 \) is a non-negative integer, and for each \( (m, n) \in \pi_1(Q) \setminus L' \) the inequality \( mr_1 + ns_1 > l_1 \) is true. Analogously, for all \( (m, k) \in L'^{\alpha} \cap R_2 \) the relation \( mr_2 + ks_2 = l_2 \) holds (\( l_2 \) is a non-negative integer) and for all \( (m, k) \in \pi_2(Q) \setminus L'^{\alpha} \) the inequality \( mr_2 + ks_2 > l_2 \) is true. Moreover, \( l_1 + l_2 > 0 \). From this point, (*) is transformed into

\[
F(\eta, \mu_1, \mu_2) \equiv P(\eta) + \sum'' \varphi_{\text{mak}}(\eta)\mu_1^{mr_1 + ns_1 - l_1} \mu_2^{mr_2 + ks_2 - l_2} = 0, \tag{8}
\]

where \( P(\eta) = \sum' \varphi_{\text{mak}}(\eta) \).\(^4\)

Let us suppose that \( \eta_0 \) is a simple non-zero root of \( P(\eta) \). Then \( F(\eta_0, 0, 0) = 0 \), \( \det F'(\eta_0, 0, 0) \neq 0 \) and according to the implicit function theorem, (8) has a single solution in a neighbourhood of the point \( (\eta_0, 0, 0) \):

\[
\eta = \eta_0 + \sum_{i+k \geq 1} \eta_{ik} \mu_1^i \mu_2^k, \tag{9}
\]

All coefficients \( \{\eta_{ik}\} \) are defined using the indefinite coefficients (IC) method. Moreover, if \( \eta_0 \in \mathbb{R}^r \) and all coefficients of (1) are real, then all \( \eta_{ik} \in \mathbb{R}^r \). Substituting (9)

\(^4\)We name the polynomial \( P(\eta) \) (respectively, the equation \( P(\eta) = 0 \)) and the defining polynomial (DP) (respectively, the defining equation (DE)) for the link \( L'^{\alpha} \).
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into the first relation of (7) and using the second and third parts of (7), we arrive at the SS (or real SS) of (1):

$$\xi = \eta_0 \lambda_1^{n_1/\eta_1} \lambda_2^{n_2/\eta_2} + \sum_{i+k \geq 1} \eta_i \lambda_1^{n_1+i/\eta_1} \lambda_2^{n_2+k/\eta_2}. \quad (10)$$

(When we write $a^{1/s}$ ($s \in \mathbb{N}$), we are taking into account only the arithmetic value of $\sqrt[n]{a}$.) Thus every simple non-zero root of the DP provides a single SS of (1) and this solution is represented in the form of the convergent series (10).

Let $\eta_0$ be a multiple root of $P(\eta)$. Then assuming $\eta = \eta_0 + u$ in (8), we obtain a new equation $F(u, \mu_1, \mu_2) = 0$ and the problem is reduced to obtaining the SS of the last equation. Standardising this equation (if required) and then repeating the same deduction as for (1'), we obtain a unique SS for every non-zero simple root of its DP. For every multiple root the procedure is repeated. If after a finite number of steps the process of obtaining multiple roots is stopped, then every SS of the problem, corresponding to $\eta_0$, is represented as a convergent series for integer or fractional non-negative powers of the parameters $\lambda_1$ and $t$, where $\lambda_1 = t \lambda_2^\sigma$ ($\sigma \geq 0$). If the process of obtaining the multiple roots is nonfinite, that is, when the SS is a multiple solution, then for a scalar equation all SS also have the structure described above. This is established in just the same way as for a scalar equation with a single small parameter (see [1]).

From Weierstrass’ preparation theorem ([6, p. 39]) it follows that for the scalar equation (1), the total number of non-zero SS is finite and equal to $m_0 - I_0$ (each solution is counted as many times as its multiplicity).5 Using all links of $D_1$ and $D_2$ and all DP, the described method gives all non-zero SS because the total number of all non-zero roots of all DP is equal to $m_0 - I_0$ (each root is counted according to its multiplicity).

**Example 1.** As an illustration of this method, we consider the problem for the scalar equation

$$\Gamma(\xi, \lambda_1, \lambda_2) \equiv \lambda_1^{5} + 2\lambda_2 \lambda_1^{2} \xi + \lambda_1^{3} \lambda_2^{2} \xi^{2} - \lambda_2^{3} \xi^{3} + 4\lambda_2 \xi^{4} + \xi^{11}$$

$$+ \sum_{m+n \geq 11} L_{mn} \lambda_1^{m} \lambda_2^{n} \xi = 0. \quad (11)$$

**Solution.** The diagram $D_1$ is represented in Figure 1 (a). It consists of 3 links: $A_1A_3$ with slope 1, $A_3A_4$ with slope 1/2 and $A_4A_6$ with slope 0. Two points $A_2$ and $A_5$ from $R_1$ are also shown in Figure 1 (a); they belong to $D_1$. The other part of $R_1$ is not situated below the straight line $(U)$ passing through points $(11;0)$ and $(0;11)$. The diagram $D_2$ is represented in Figure 1 (b). It consists of 4 links: $B_1B_3$ with slope $-1$,

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5The number $I_0$ is equal to the multiplicity of the zero root of the DP for the left link of $D_2$. 
$B_1 B_4$ with slope 0, $B_4 B_5$ with slope 1 and $B_3 B_6$ with slope $1/2$. The diagram $D_2$ is not standard. For standardising we use (4). For the point $B_4$ and $B_5$, the right side of (4) is equal to 2; we have $s_0 = 2$. Using (6), Figures 1 (a) and (b), we define $s = 1$. Thus $s = \max(2; 1)$ and by the substitution $\lambda_1 = \mu_1 \lambda_2^2$, (11) is standardised. Further, we construct the diagram $D_2$ (see Figure 1 (c)). It consists of 4 links: $\tilde{B}_1 \tilde{B}_3$ (the slope is equal to 1, the DP is $\eta^2 + 2\eta + 1$, its roots are $-1, -1$ (double root)); $\tilde{B}_3 \tilde{B}_4$ (the slope is 1, the DP is $-\eta^2 + \eta^2$, its non-zero roots $a_k (k = 1, \ldots, 6)$ are the system of all roots $\sqrt{1}$, each root is simple); $\tilde{B}_4 \tilde{B}_5$ (the slope is 1, the DP is $4\eta^3 - \eta^3$; the unique non-zero root of the DP is $b_0 = 1/4$ (simple)); $\tilde{B}_5 \tilde{B}_6$ (the slope is 1/2, the DP is $\eta^{11} + 4 \eta^3$, the non-zero roots of this polynomial are simple, they are $c_0 (1) = 2i$, $c_0 (2) = -2i$).

We obtain for $(A_3 A_4, \tilde{B}_4 \tilde{B}_5)$, six SS: $\xi^{(k)} = d_0^{(k)} \mu_1^{1/2} \lambda_2 + \sum_{m+n \geq 1} c_{mn}^{(k)} \mu_1^{(m+1)/2} \lambda_2^{n+1}$, $k = 1, \ldots, 6$; for $(A_4 A_6, \tilde{B}_4 \tilde{B}_5)$, one SS: $\xi = b_0 \lambda_2 + \sum_{m+n \geq 1} b_{mn} \mu_1^{(m+1)/2} \lambda_2^{n+1}$; for $(A_4 A_6, \tilde{B}_5 \tilde{B}_6)$, two SS: $\xi^{(v)} = c_0^{(v)} \mu_1^{1/2} \lambda_2 + \sum_{m+n \geq 1} c_{mn}^{(v)} \mu_1^{(m+1)/2} \lambda_2^{n+1}$, $v = 1, 2$, where
For obtaining the SS corresponding to \((A_1A_3, B_1B_3)\), we use the substitution \(\xi = (v - 1)\mu_1\lambda_2\). We get the equation

\[ v^2 - \mu_1^3 + \cdots = 0, \tag{12} \]

where the left-hand side does not include the terms of the form \(a\lambda_1^k\lambda_2^2\) and \(b\lambda_1^2\lambda_2^v\) \((k\) is a natural number). Equation \((12)\) is standard. The diagram \(D_1\) for \((12)\) consists of a unique link joined to the points \((0; 0)\) and \((2; 0)\) \((\text{the slope is } 3 = 2)\); the diagram \(D_2\) also consists of one link denoted as \(C_1C_2\) \((C_1.0; 0)\) and \((C_2.2; 0)\)). Both roots of the DP for \(C_1C_2\) are simple and equal to 1 and \(-1\). Equation \((12)\) has two SS; they are of the form

\[ v = \mu_1\lambda_2^2 \tag{13} \]

Finally, the pair of links \((A_1A_3, B_1B_3)\) provides two SS: \(\xi = -\mu_1\lambda_2 (1 - v^{(k)})\), where \(k = 1, 2\), and \(\lambda_1 = \mu_1\lambda_2^2\).

### 3. Critical situation

If \((1)\) is vectorial \((r > 1)\), then the described method may not always obtain all the SS. For example, the two-dimensional equation

\[ \varphi_{200}(\xi) + \varphi_{300}(\xi) + \varphi_{010}\lambda_1 + \varphi_{001}\lambda_2 = 0, \tag{13'} \]

where \(\xi = (\xi_1, \xi_2)\), \(\varphi_{200}(\xi) = \{3\xi_1^2 - \xi_2^2; 0\}\), \(\varphi_{300}(\xi) = \{0; (\xi_1 - \xi_2)^3\}\), \(\varphi_{010} = \varphi_{001} = \{1; 8\}\), has six SS. However, if we use Newton’s diagram method, we get the following. The diagram \(D_1\) for \((13)\) consists of one link with ends \((0; 0)\) and \((2; 0)\) \((p^{(1)} = 0)\); the diagram \(D_2\) consists of a unique link \(AB\) \((A.0; 1), B.(2; 0)\) with slope \(1/2\). The substitution \(\lambda_1 = \mu_1\lambda_2\) reduces \((13)\) to the standard case:

\[ \tilde{\varphi}_{200}(\xi) + \tilde{\varphi}_{300}(\xi) + \tilde{\varphi}_{010}\mu_1\lambda_2 + \tilde{\varphi}_{001}\lambda_2 = 0, \tag{13''} \]

where \(\tilde{\varphi}_{010} = \varphi_{010}\). The view of \(\tilde{D}_2\) is the same as \(D_2\), but \(\tilde{D}_2\) is a standard diagram. The DP for \(AB\) of \(\tilde{D}_2\) is \(\{3\eta_1^2 - \eta_2^2 + 1; 8\}\). This polynomial is not solvable in \(\mathbb{C}\) and therefore Newton’s diagram method gives nothing.

In order that the total number of SS be finite for the vectorial equation \((1)\) and all the SS be obtainable by using Newton’s diagram method, the following condition is sufficient: all the fields, corresponding to the right ends of the links of \(D_2\) \((\text{for all stages of using this method})\), are non-degenerate in \(\mathbb{C}'\). The above condition is a corollary from the following theorem \([2]\).
\[ Q_i(x_1, \ldots, x_n) + \Omega_i(x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, n, \]  
\[ (14) \]

be an algebraic system with complex coefficients, where \( Q_i \) is a homogeneous polynomial of degree \( n_i \) (\( n_i \geq 1 \)) and \( \Omega_i \) is a zero polynomial or a polynomial of degree \( m_i < n_i \). Then the following is true: if the field \( \{Q_1, \ldots, Q_n\} \) is non-degenerate in \( \mathbb{C}^n \), then \( (14) \) is regular in \( \mathbb{C}^n \), that is, \( (14) \) is solvable in \( \mathbb{C}^n \) and the total number of its solutions is finite.

The non-degenerating condition for polynomial fields in \( \mathbb{C}^n \) is also described in [2]. For \( (13) \) the sufficient condition for using Newton’s diagram method is infringed, because the field \( \varphi_{200} \) is degenerate (the point \( (2, 0, 0) \) provides the right end of the link \( AB \in \tilde{D}_2 \)). Moreover, the DE for \( AB \) is not solvable in \( \mathbb{C}^2 \). The point of \( Q \) providing the right end of \( \tilde{D}_2 \) is called a critical point of \( Q \) if the DE for this link is not regular in \( \mathbb{C}^2 \). In the case when \( Q_0 \) has bad points, we can offer two methods. The first one is the combination of Newton’s diagram method and a new method called the method of removal of the bad corner points. The main idea is the following: if \( (m, n, k) \in Q \) is a bad point, we remove it from \( Q \) and construct for the rest of \( Q \) the diagrams \( D_1 \) and \( \tilde{D}_2 \). In this situation we use the diagram only to define the possible exponents for the small parameters. For obtaining the coefficients of the expansion of the SS, we use a substitution in the form

\[ \xi = \eta \mu_1^p \lambda_2^{p_1} + x(\mu_1, \lambda_2) \mu_1^p \lambda_2^{p_1}, \]
\[ (15) \]

where \( \eta = (\eta_1, \ldots, \eta_k), x = (x_1, \ldots, x_n) \) and \( x(\mu_1, \lambda_2) \to 0 \) as \( (\mu_1, \lambda_2) \to (0, 0) \).

To illustrate this method we choose \( (13) \). The lattice for \( (13) \) is

\[ Q = \{ (2, 0, 0), (3, 0, 0), (0, 1, 0), (0, 0, 1) \}. \]

We showed before that \( (2, 0, 0) \) is a bad point of \( Q \). According to this method, we remove \( (2, 0, 0) \) from \( Q \) and construct the diagrams \( D_1 \) and \( \tilde{D}_2 \) for the lattice \{ \( (3, 0, 0), (0, 1, 0), (0, 0, 1) \) \}: the diagram \( D_1 \) consists of a unique link with slope 0; the diagram \( \tilde{D}_2 \) also consists of a unique link joining \( (0, 1) \) and \( (3, 0) \) with slope 1/3. Using the substitution \( \lambda_1 = \mu_1 \lambda_2 \), we get the standard diagram \( \tilde{D}_2 \). The view of \( \tilde{D}_2 \) is the same as \( D_2 \). Thus the possible exponents are 0 and 1/3.

Now, for obtaining the SS of \( (13) \), we use \( (15) \), where \( p_1 = 0 \) and \( p_\ell, r = 1/3 \). We get the system

\[ -x_2^2 + 3\eta_1^2 - \eta_2^2 + 6\eta_1 x_1 - 2\eta_2 x_2 + 3x_1^2 + \mu_1 \lambda_2^{1/3} + \lambda_2^{1/3} = 0, \]
\[ [(\eta_1 - \eta_2) + (x_1 - x_2)]^3 + 8 + 8\mu_1 = 0. \]
\[ (16) \]

\*When we say “Newton’s diagram” for \( \pi_i(Q) \) or \( \pi_2(Q) \), we have in view only their decreasing branches.
Passing to the limit as $(\mu_1, \lambda_2) \to (0, 0)$ in (16), we obtain the DE for $\eta$:

$$3\eta_1^2 - \eta_2^2 = 0, \quad (\eta_1 - \eta_2)^3 + 8 = 0. \quad (17)$$

The system (17) is regular and has exactly six simple roots $\eta' = (\eta_1^{(i)}, \eta_2^{(i)}), i = 1, \ldots, 6$ (only two of them are real). For obtaining $x(\mu_1, \lambda_2)$, we substitute $\lambda_2 = \mu_2^3$ into (16) and for each root $\eta'$ obtain the system

$$6\tilde{\eta}_1^{(i)} x_1 - 2\tilde{\eta}_2^{(i)} x_2 + 3x_1^2 - x_2^2 + \mu_1 \mu_2 + \mu_2 = 0,$$

$$3\tilde{\eta}_2^2 (x_1 - x_2) + 3\tilde{\eta}_1 (x_1 - x_2)^2 + (x_1 - x_2)^3 + 8\mu_1 = 0. \quad (18)$$

where $\tilde{\eta}_i = \eta_1^{(i)} - \eta_2^{(i)}$. Again using the method for each (18), we obtain the first term of the expressions for $x$. However, the roots $\eta'$ are simple and we can use the IC method.

Consequently, the problem for (13) has exactly six SS and they have the view

$$\xi^i = \eta' \mu_2 + \mu_2 \sum_{k=1}^{\infty} a^k \mu_1^k \mu_2^2, \quad (\lambda_1 = \mu_1 \mu_2^3, \lambda_2 = \mu_2^3; i = 1, \ldots, 6).$$

Two SS among them belong to $\mathbb{R}^2$.

The second method, called the NDE method, is a combination of Newton’s diagram method for scalar equations and the method of elimination. In the case of a single scalar small parameter it was described in [4, 6]. For several small parameters the way to obtain the SS is analogous. According to the NDE method, we get the scalar equations for each component of the unknown variables and use Newton’s diagram for each equation. The main difference is in the following: in the case of a single parameter we use the classical Newton’s diagram method, in the case of several parameters we use the method described in this article.

Note here that the necessary and sufficient condition for the general system with a single small parameter to be regular is presented in [4, 6]. In the case of several parameters the condition is the same.

### 4. Case of $n$ $(n > 2)$ parameters

In this section, Newton’s diagram method is extended to the general case, when the equation includes more than two small parameters. We investigate the equation

$$\Gamma(\xi, \lambda_1, \ldots, \lambda_n) \equiv \sum_{m+k_1+\ldots+k_n=1}^{\infty} \varphi_{m_k1} (\xi) \lambda_1^{k_1} \ldots \lambda_n^{k_n} = 0, \quad (19)$$
where \(\lambda_1, \ldots, \lambda_n\) are small parameters \((n > 2)\), \(\varphi_m(\xi)\) is a vectorial homogeneous polynomial of power \(m\) or a zero polynomial. Let us assume that \(\varphi_{m-0}(0) = 0\) at \(m < m_0\) and \(\varphi_{m-0}(0) \neq 0\). We are studying the problem about the SS \(\xi(\lambda_1, \ldots, \lambda_n)\) of (19) for \((\lambda_1, \ldots, \lambda_n)\) near \((0, \ldots, 0)\) and near the hyperplane \(\lambda_1 = 0\) without \(\lambda_2 = \cdots = \lambda_n = 0\). Let us denote \(\mathcal{Q} = \{(m, k_1, \ldots, k_n) : \varphi_{m k_1 \cdots k_n}(\xi) \neq 0\}\), \(\mathcal{Q}_0 = \{(m, k_1, \ldots, k_n) \in \mathcal{Q} : m \leq m_0\}\), \(R_i = \pi_i(\mathcal{Q}_0), i = 1, \ldots, 6\), where \(\pi_i\) is the orthogonal projection of \(\mathbb{R}^{n+1}\) on the coordinate plane \(OMK_i\). On the plane \(OMK_1\), we construct Newton’s diagram \(D_1 = \{L_{i-1}^{(i)} \mid i_1 = 1, \ldots, \alpha_1\}\) for \(R_1\). After that we construct, on the plane \(OMK_2\), Newton’s diagram for each lattice \(\pi_2(\pi_i^{-1}(L_i^{(i)}) \cap \mathcal{Q}_0), i_1 = 1, \ldots, \alpha_1\). We get a diagram \(D_2 = \{L_{i-1}^{(i)} \mid i_2 = 1, \ldots, \alpha_2^{(i_1)}, i_1 = 1, \ldots, \alpha_1\}\). If \(D_2\) is Newton’s diagram for the lattice \(R_2\) and it has no points from \(R_2 \setminus \bigcup_i \pi_2(\pi_i^{-1}(L_i^{(i)}) \cap Q_0)\), \(i_1 = 1, \ldots, \alpha_1\). If \(D_2\) is a standard diagram for \(R_2\) without additional points, substituting \(\lambda_1 = \mu_1 \lambda_2^{\alpha_1}\) \((s_1)\) is chosen according to (4) and (6), and Notes 1 and 2) and only after that we use \(D_2\) and construct \(D_3\). This procedure is continued until the diagram \(D_n\) comes onto the plane \(OMK_n\). If \(D_n\) is not a standard diagram, we use the substitution \(\lambda_{n+1} = \mu_{n+1} \lambda_n^{\alpha_n}\) and reduce \(D_n\) to the standard form. For the last diagram it is assumed that the slopes of its links make up only a non-decreasing sequence and these links can include points of \(R_n\) with prototypes not belonging to \(D_{n-1}\).

For example, we consider the equation

\[
\varphi_{0202}(\xi)\lambda_2^2\lambda_3 + \varphi_{2111}(\xi)\lambda_1\lambda_2\lambda_3 + \varphi_{4022}(\xi)\lambda_2^2\lambda_3^2 + \varphi_{0101}(\xi)\lambda_1\lambda_3
+ \varphi_{8001}(\xi)\lambda_3 + \varphi_{10:000}(\xi) + \sum_{m+k_1 \geq 11} \varphi_{m k_1 k_2 k_3}(\xi)\lambda_1^k_1\lambda_2^k_2\lambda_3^k_3 = 0 \quad (\star)
\]

with three parameters \(\lambda_1^k_1, \lambda_2^k_2, \lambda_3^k_3\). We get the following.

The diagram \(D_1\) (Figure 2 (a)) consists of two links: \(A_1A_3\) with slope 1/2 and \(A_2A_4\) with slope 0. The points \(A_2, A_4, A_5\) from \(R_1\) also belong to \(D_1\). The other points of \(R_1\) are situated on the right of the line connecting the points \((0; 10)\) and \((10; 0)\).

The diagram \(D_2\) is presented in Figure 2 (b). Using the substitution \(\lambda_1 = \mu_1 \lambda_2^3\), the diagram \(D_2\) is reduced to Newton’s diagram \(D_2\) without additional points. The diagram \(D_2\) (Figure 2 (c)) consists of three links: \(B_1B_3\) with slope 1, \(B_3B_2\) with slope 1/2 and \(B_2B_4\) with slope 0. The other points of \(R_2\) are situated on the right of the line connecting the points \((0; 30)\) and \((10; 0)\).

The diagram \(D_3\) is represented in Figure 2 (d). The substitution \(\lambda_2 = \mu_2 \lambda_3^2\) reduces \(D_3\) to the standard diagram \(D_3\) (Figure 3).

Let us suppose that \(D_4\) for (19) is Newton’s diagram; \(D_2, \ldots, D_{n-1}\) are Newton’s diagrams without additional points and \(D_n\) is a standard diagram, that is, we have the

\[\ldots\]

3We name this diagram Newton’s diagram without additional points.
standard case.\footnote{If not, we, at first, reduce (19) to the standard case.} We write (19) in the form
\[
\sum' \varphi_{m_{k_1-k_n}}(\xi) \lambda_1^{k_1} \cdots \lambda_n^{k_n} + \sum'' \varphi_{m_{k_1-k_n}}(\xi) \lambda_1^{k_1} \cdots \lambda_n^{k_n} = 0, \tag{19'}
\]
where $\sum'$ denotes the sum of all non-zero terms which satisfy the condition:
\[
(m, k_1) \in L^{i_1}, \quad (m, k_2) \in L^{i_1i_2}, \quad \ldots, \quad (m, k_n) \in L^{i_1\cdots i_n}. 
\]
Here $L^{i}$ is any link of $D_1$, $L^{i_1i_2}$ is any link of $D_2$ corresponding to $L^{i}$ and etc.; at last, $L^{i_1\cdots i_n}$ is any link of $D_n$ corresponding to the link $L^{i_1\cdots i_n}$. The sum $\sum''$ denotes the sum of all other terms of (19). We assume also that $r_k/s_k ((r_k, s_k) = 1)$ is the slope of $L^{i_1\cdots i_k}, k = 1, \ldots, n$.

Let us make the substitutions $\xi = \eta \mu_1^{(i)} \cdots \mu_n^{(i)}, \lambda_1 = \mu_1^{(i)}, \ldots, \lambda_n = \mu_n^{(i)}$ in (19'). Then the equation is reduced to the form
\[
P(\eta) + \sum' \varphi_{m_{k_1-k_n}}(\eta) \mu_1^{v_1} \cdots \mu_n^{v_n} = 0,
\]
where $v_1, \ldots, v_n$ are non-negative numbers, $v_1 + \cdots + v_n > 0$ and $P(\eta) = \sum' \varphi_{m_{k_1-k_n}}(\eta)$ is the DP for the link $L^{i_1\cdots i_n}$. The further operations, reasons and conclusions are analogous to the case of two small parameters.

Finally we can make some additional remarks.
**Remark 1.** The stated method allows us to find SS defined in a neighbourhood of a hypersurface \( \lambda_n = g(\lambda_1, \ldots, \lambda_{n-1}) \), where \( g \) is an analytical function relative to \( \lambda_1^{p_1}, \ldots, \lambda_{n-1}^{p_{n-1}}, \) \((p_1, \ldots, p_{n-1} \) are natural numbers) and \( g \to 0 \) as \((\lambda_1, \ldots, \lambda_{n-1}) \to (0, \ldots, 0)\).

Indeed, letting \( \lambda_1 = \mu_1^{p_1}, \ldots, \lambda_{n-1} = \mu_{n-1}^{p_{n-1}}, \lambda_n = g + \mu_1 \) in (19), we reduce this problem to the problem about the SS for \((\mu_1, \ldots, \mu_n)\) defined in a neighbourhood of the hyperplane \( \mu_1 = 0 \).

**Remark 2.** If (19) is represented by the standard case and all roots of all the DP are simple, then all the SS of (19) are defined in some full neighbourhood of \( \lambda_1 = \cdots = \lambda_n = 0 \).

If the diagrams \( D_i \) \((i = 1, \ldots, k; k < n)\) are Newton’s diagrams without additional points and all root of the DP are simple, then the SS are defined in a neighbourhood of the hyperplane \( \lambda_1 = \cdots = \lambda_k = 0 \). In this situation the number \( k \) is called the coefficient of standardisation. This coefficient is important in some applications.

**References**


