TWO-POINT FORMULAE OF EULER TYPE

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Abstract

An analysis is made of quadrature via two-point formulae when the integrand is Lipschitz or of bounded variation. The error estimates are shown to be as good as those found in recent studies using Simpson (three-point) formulae.

1. Introduction and preliminaries

The simplest quadrature rule of open type is based on the well-known midpoint formula

\[ \int_a^b f(t) \, dt = (b - a) f \left( \frac{a + b}{2} \right) + \frac{(b - a)^3}{24} f''(\xi), \quad (1.1) \]

where \( a < \xi < b \) (see [3, p. 71]). Another quadrature rule of this type is based on the two-point formula

\[ \int_a^b f(t) \, dt = \frac{b - a}{2} \left[ f \left( \frac{2a + b}{3} \right) + f \left( \frac{a + 2b}{3} \right) \right] + \frac{(b - a)^3}{36} f''(\eta), \quad (1.2) \]

where \( a < \eta < b \) (see [3, p. 70]). Both formulae apply provided \( f : [a, b] \to \mathbb{R} \) is in the class \( C^2[a, b] \).

For a convex function \( f \in C^2[a, b] \) we have \( f''(\xi) \geq 0 \), so a simple consequence of (1.1) for such functions is the Hadamard inequality

\[ \frac{1}{b - a} \int_a^b f(t) \, dt \geq f \left( \frac{a + b}{2} \right). \quad (1.3) \]
By the same argument, (1.2) yields

$$\frac{1}{b-a} \int_a^b f(t) \, dt \geq \frac{1}{2} \left[ f \left( \frac{2a+b}{3} \right) + f \left( \frac{a+2b}{3} \right) \right]$$

(1.4)

for any convex function $f \in C^2[a, b]$.

Inequality (1.4) is tighter than (1.3) for $f$ convex, since

$$\frac{1}{2} \left[ f \left( \frac{2a+b}{3} \right) + f \left( \frac{a+2b}{3} \right) \right] \geq f \left( \frac{1}{2} \cdot \frac{2a+b}{3} + \frac{1}{2} \cdot \frac{a+2b}{3} \right) = f \left( \frac{a+b}{2} \right).$$

However, we can obtain (1.4) by using (1.3) on subintervals. The latter inequality provides

$$\int_a^b f(t) \, dt = \int_a^{(a+b)/2} f(t) \, dt + \int_{(a+b)/2}^b f(t) \, dt \geq \frac{b-a}{2} \left[ f \left( \frac{3a+b}{4} \right) + f \left( \frac{a+3b}{4} \right) \right].$$

(1.5)

On the other hand, a convex function $f : [a, b] \rightarrow \mathbb{R}$ satisfies

$$f(x+z) - f(x) \leq f(y+z) - f(y)$$

whenever $x$, $y$ and $z$ are such that $x, x+z, y, y+z \in [a, b]$ with $x \leq y$ and $z \geq 0$ (see [11, p. 3]). In particular, the choices $x = (3a+b)/4$, $y = (a+2b)/3$ and $z = (b-a)/12$ yield

$$f \left( \frac{2a+b}{3} \right) - f \left( \frac{3a+b}{4} \right) \leq f \left( \frac{a+3b}{4} \right) - f \left( \frac{a+2b}{3} \right),$$

that is,

$$f \left( \frac{2a+b}{3} \right) + f \left( \frac{a+2b}{3} \right) \leq f \left( \frac{3a+b}{4} \right) + f \left( \frac{a+3b}{4} \right).$$

Combining this with (1.5) supplies (1.4).

Midpoint formulae of Euler type, based on (1.1), were treated recently in [4]. In this paper we consider similar results related to the two-point formula (1.2).

The fundamental ingredients in our analysis are the same, namely the two identities

$$f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt + T_n(x) + \mathcal{R}_n^1(x)$$

(1.6)

and

$$f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt + T_{n-1}(x) + \mathcal{R}_n^2(x),$$

(1.7)
which may conveniently be referred to as the extended Euler formulae and which were established recently in [5]. Here $T_0(x) = 0$ and

$$T_m(x) = \sum_{k=1}^{m} \frac{(b - a)^{k-1}}{k!} B_k \left( \frac{x - a}{b - a} \right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right]$$  \quad (1.8)

for $m \geq 1$, while

$$R^1_n(x) = -\frac{(b - a)^{n-1}}{n!} \int_{[a,b]} B_n^* \left( \frac{x - t}{b - a} \right) df^{(n-1)}(t)$$

and

$$R^2_n(x) = -\frac{(b - a)^{n-1}}{n!} \int_{[a,b]} \left[ B_n^* \left( \frac{x - t}{b - a} \right) - B_n \left( \frac{x - a}{b - a} \right) \right] df^{(n-1)}(t).$$

We write $\int_{[a,b]} g(t) d\varphi(t)$ here, as throughout the paper, to denote the Riemann-Stieltjes integral of $g$ with respect to a function $\varphi : [a, b] \to \mathbb{R}$ of bounded variation and $\int_{[a,b]} g(t) \, dt$ for the Riemann integral. The identities (1.6) and (1.7) extend the well-known formula for the expansion of a function in terms of Bernoulli polynomials [10, p. 17]. They hold for every function $f : [a, b] \to \mathbb{R}$ such that $f^{(n-1)}$ is continuous and of bounded variation on $[a, b]$ for some $n \geq 1$ and for every $x \in [a, b]$. The functions $B_k(t)$ are the Bernoulli polynomials, $B_k = B_k(0)$ the Bernoulli numbers and $B_k^*(t)$ ($k \geq 0$) are functions of period 1 related to the Bernoulli polynomials via

$$B_k^*(t) = B_k(t), \quad \text{for } 0 \leq t < 1,$$

$$B_k^*(t + 1) = B_k^*(t), \quad \text{for } t \in \mathbb{R}.$$  

The Bernoulli polynomials $B_k(t)$ ($k \geq 0$) are uniquely determined by the identities

$$B_k(t) = kB_{k-1}(t), \quad k \geq 1; \quad B_0(t) = 1$$  \quad (1.9)

and

$$B_k(t + 1) - B_k(t) = kt^{k-1}, \quad k \geq 0.$$  \quad (1.10)

For further details on the Bernoulli polynomials and the Bernoulli numbers, see for example [1] or [2]. We have

$$B_0(t) = 1, \quad B_1(t) = -t - 1/2, \quad B_2(t) = t^2 - t + 1/6, \quad B_3(t) = t^3 - 3t^2/2 + t/2,$$  \quad (1.11)

so that $B_0^*(t) = 1$ and $B_1^*(t)$ has a jump of $-1$ at each integer. From (1.10) it follows that $B_k(1) = B_k(0) = B_k$ for $k \geq 2$, so that $B_k^*(t)$ is continuous for $k \geq 2$. Moreover, using (1.9) we get

$$B_k^* = kB_{k-1}^*, \quad k \geq 1$$  \quad (1.12)

and this holds for every $t \in \mathbb{R}$ when $k \geq 3$, and for every $t \in \mathbb{R} \setminus \mathbb{Z}$ when $k = 1, 2$. 

As in [4], our analysis hangs on detailed properties of the Bernoulli polynomials. The analysis is effected via two families \((F_k)_{k \geq 1}\) and \((G_k)_{k \geq 1}\) of auxiliary functions. The basic idea of the two-point approach is outlined in Section 2 and centres on two two-point formulae. In Section 3 we develop the requisite results for the auxiliary functions and in Section 4 use these to determine error estimates when integrals are approximated by our two-point formulae. We consider integrands which are either of bounded variation or possess a Lipschitz property. We find that the error estimates for our current two-point procedures are as good as those obtained recently for three-point (Simpson) procedures (see [7, 8, 6, 9]). Finally in Section 5 we make corresponding estimates when the domain of integration is given a general uniform partition and the two-point formulae are repeated for quadrature.

2. Generalisations of the two-point formula

For \(k \geq 1\), define the functions \(G_k(t)\) and \(F_k(t)\) by
\[
G_k(t) := B_k^*(1/3 - t) + B_k^*(2/3 - t), \quad t \in \mathbb{R}
\]
and
\[
F_k(t) := G_k(t) - \tilde{B}_k, \quad t \in \mathbb{R},
\]
where
\[
\tilde{B}_k := G_k(0) = B_k(1/3) + B_k(2/3), \quad k \geq 1.
\]
The functions \(G_k(t)\) and \(F_k(t)\) are of period 1 and continuous for \(k \geq 2\) and so are determined by their behaviour on \([0, 1]\). This we investigate in the next section.

Let \(f : [a, b] \to \mathbb{R}\) be such that \(f^{(n-1)}\) exists on \([a, b]\) for some \(n \geq 1\). We introduce the notation
\[
M(a, b) := \frac{b - a}{2} \left[ f \left( \frac{2a + b}{3} \right) + f \left( \frac{a + 2b}{3} \right) \right].
\]
Further, define
\[
\tilde{T}_0(a, b) := 0
\]
and
\[
\tilde{T}_m(a, b) := \frac{b - a}{2} \left[ T_m \left( \frac{2a + b}{3} \right) + T_m \left( \frac{a + 2b}{3} \right) \right]
\]
for \(1 \leq m \leq n\), where \(T_m(x)\) is given by (1.8). Then
\[
\tilde{T}_m(a, b) = \frac{1}{2} \sum_{k=1}^{m} \frac{(b - a)^k}{k!} \tilde{B}_k \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right].
\]
In the theorem below we establish two formulae which we term two-point formulae of Euler type and which play a key role in this paper.

**THEOREM 1.** Let \( f : [a, b] \to \mathbb{R} \) be such that \( f^{(n-1)} \) is continuous and of bounded variation on \([a, b]\) for some \( n \geq 1 \). Then

\[
\int_a^b f(t) \, dt = M(a, b) - \tilde{T}_n(a, b) + \tilde{R}_n^1(a, b),
\]

where

\[
\tilde{R}_n^1(a, b) = \frac{(b-a)^n}{2(n!)} \int_{[a,b]} G_n \left( \frac{t-a}{b-a} \right) \, df^{(n-1)}(t).
\]

Also

\[
\int_a^b f(t) \, dt = M(a, b) - \tilde{T}_{n-1}(a, b) + \tilde{R}_n^2(a, b),
\]

where

\[
\tilde{R}_n^2(a, b) = \frac{(b-a)^n}{2(n!)} \int_{[a,b]} F_n \left( \frac{t-a}{b-a} \right) \, df^{(n-1)}(t).
\]

**PROOF.** Put \( x = \frac{(2a+b)}{3}, \frac{(a+2b)}{3} \) in (1.6), multiply the two resultant formulae by \( (b-a)/2 \) and add. This produces (2.3). Formula (2.4) is obtained from (1.7) by the same procedure.

**REMARK 1.** Suppose that \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(n)} \) exists and is integrable on \([a, b]\) for some \( n \geq 1 \). In this case (2.3) holds with

\[
\tilde{R}_n^1(a, b) = \frac{(b-a)^n}{2(n!)} \int_a^b G_n \left( \frac{t-a}{b-a} \right) f^{(n)}(t) \, dt,
\]

while (2.4) holds with

\[
\tilde{R}_n^2(a, b) = \frac{(b-a)^n}{2(n!)} \int_a^b F_n \left( \frac{t-a}{b-a} \right) f^{(n)}(t) \, dt.
\]

By direct calculation we get \( \tilde{B}_1 = 0, \tilde{B}_2 = -1/9, \tilde{B}_3 = 0 \). This implies, by (2.2), that

\[
\tilde{T}_0(a, b) = \tilde{T}_1(a, b) = 0, \quad \tilde{T}_2(a, b) = -\frac{(b-a)^2}{36} \left[ f''(b) - f'(a) \right].
\]

Also

\[
G_1(t) = F_1(t) = \begin{cases} 
-2t, & 0 \leq t \leq 1/3; \\
-2t + 1, & 1/3 < t \leq 2/3; \\
-2t + 2, & 2/3 < t \leq 1,
\end{cases}
\]
\[ G_2(t) = \begin{cases} 2t^2 - 1/9, & 0 \leq t \leq 1/3; \\ 2t^2 - 2t + 5/9, & 1/3 < t \leq 2/3; \\ 2t^2 - 4t + 17/9, & 2/3 \leq t \leq 1, \end{cases} \quad (2.7) \]

\[ F_2(t) = \begin{cases} 2t, & 0 \leq t \leq 1/3; \\ 2t^2 - 2t + 2/3, & 1/3 < t \leq 2/3; \\ 2t^2 - 4t + 2, & 2/3 \leq t \leq 1, \end{cases} \quad (2.8) \]

and

\[ F_3(t) = G_3(t) = \begin{cases} -2t^3 + t/3, & 0 \leq t \leq 1/3; \\ -2t^3 + 3t^2 - 5t/3 + 1/3, & 1/3 < t \leq 2/3; \\ -2t^3 + 6t^2 - 17t/3 + 5/3, & 2/3 \leq t \leq 1. \end{cases} \quad (2.9) \]

Applying (2.4) with \( n = 1, 2 \) yields the identities

\[
\int_a^b f(t) \, dt - M(a, b) = \frac{b - a}{2} \int_{[a,b]} F_1 \left( \frac{t - a}{b - a} \right) \, df(t) 
\]

\[
= \frac{(b - a)^2}{4} \int_{[a,b]} F_2 \left( \frac{t - a}{b - a} \right) \, df'(t).
\]

Similarly, (2.4) with \( n = 3, 4 \) generates the identities

\[
\int_a^b f(t) \, dt - M(a, b) - \frac{(b - a)^2}{36} \left[ f'(b) - f'(a) \right]
\]

\[
= \frac{(b - a)^3}{12} \int_{[a,b]} F_3 \left( \frac{t - a}{b - a} \right) \, df''(t) = \frac{(b - a)^4}{48} \int_{[a,b]} F_4 \left( \frac{t - a}{b - a} \right) \, df'''(t).
\]

### 3. The auxiliary functions

To proceed to error estimates, we need some properties of the functions \( G_k(t) \) and \( F_k(t) \). As noted earlier, it is enough to know these on \([0, 1]\).

The Bernoulli polynomials of even order are symmetric and those of odd order skew-symmetric about 1/2, that is,

\[ B_k(1 - t) = (-1)^k B_k(t), \quad 0 \leq t \leq 1, \quad k \geq 1 \quad (3.1) \]

(see [1, 23.1.8]). Setting \( t = 1/3 \) gives \( B_k(2/3) = (-1)^k B_k(1/3) \), so that

\[ \tilde{B}_k = B_k(1/3) + B_k(2/3) = \left[ 1 + (-1)^k \right] B_k(1/3) \quad (k \geq 1), \]

which implies \( \tilde{B}_{2k-1} = 0, \tilde{B}_{2k} = 2B_{2k}(1/3) \) \( (k \geq 1) \). Also

\[ B_{2k}(1/3) = -2^{-1} \left( 1 - 3^{-1-2k} \right) B_{2k}, \quad k \geq 1, \quad (3.2) \]
(see [1, 23.1.23]), which gives
\[ \tilde{B}_{2k-1} = 0, \quad \tilde{B}_{2k} = -(1 - 3^{1-2k})B_{2k}, \quad k \geq 1. \]  
(3.3)

Now by (3.3) we have
\[ F_{2k-1}(t) = G_{2k-1}(t), \quad k \geq 1 \]  
(3.4)
and
\[ F_{2k}(t) = G_{2k}(t) + (1 - 3^{1-2k})B_{2k}, \quad k \geq 1. \]  
(3.5)

Further, the points 0 and 1 are zeros of \( F_n(t) \), that is, \( F_n(0) = F_n(1) = 0 \) \( (n \geq 1) \). As we shall see below, they are the only zeros of \( F_n(t) \) for \( n = 2k \) \( (k \geq 1) \). Also, using (3.1) again, we get \( G_n(1/2) = B_n(5/6) + B_n(1/6) = [(-1)^n + 1]B_n(1/6) \). Hence for \( n = 2k - 1 \) \( (k \geq 1) \) we have \( F_{2k-1}(1/2) = G_{2k-1}(1/2) = 0 \).

We shall see that 0, 1/2 and 1 are the only zeros of \( F_{2k-1}(t) = G_{2k-1}(t) \) for \( k \geq 1 \). Also note that for \( n = 2k \) \( (k \geq 1) \) we have
\[ G_{2k}(0) = G_{2k}(1) = \tilde{B}_{2k} = -(1 - 3^{1-2k})B_{2k}. \]  
(3.6)
Using [1, 23.1.24] \( B_{2k}(1/6) = B_{2k}(5/6) = 2^{\frac{1}{2}}(1 - 2^{1-2k})(1 - 3^{1-2k})B_{2k}, \ k \geq 1, \) we get
\[ G_{2k}(1/2) = 2B_{2k}(1/6) = (1 - 2^{1-2k})(1 - 3^{1-2k})B_{2k} \quad (k \geq 1), \]  
(3.7)
while \( F_{2k}(1/2) = G_{2k}(1/2) - \tilde{B}_{2k} = 2(1 - 2^{1-2k})(1 - 3^{1-2k})B_{2k}, \ k \geq 1. \)

**Lemma 1.** For \( n \geq 2 \) we have \( G_n(1-t) = (-1)^n G_n(t) \) and \( F_n(1-t) = (-1)^n F_n(t) \), \( 0 \leq t \leq 1 \).

**Proof.** Since \( B_n^*(t) \) is of period 1 and continuous for \( n \geq 2 \), we have for \( n \geq 2 \) and \( 0 \leq t \leq 1 \) that
\[
G_n(t) = B_n^*(1/3 - t) + B_n^*(2/3 - t) \\
= \begin{cases} 
B_n(1/3 - t) + B_n(2/3 - t), & 0 \leq t \leq 1/3; \\
B_n(4/3 - t) + B_n(2/3 - t), & 1/3 < t \leq 2/3; \\
B_n(4/3 - t) + B_n(5/3 - t), & 2/3 < t \leq 1 
\end{cases}
\]
and
\[
G_n(1-t) = B_n^*(-2/3 + t) + B_n^*(-1/3 + t) \\
= \begin{cases} 
B_n(1/3 + t) + B_n(2/3 + t), & 0 \leq t < 1/3; \\
B_n(1/3 + t) + B_n(-1/3 + t), & 1/3 \leq t < 2/3; \\
B_n(-2/3 + t) + B_n(-1/3 + t), & 2/3 \leq t \leq 1. 
\end{cases}
\]
Further, using (3.1) we get

\[ G_n(1 - t) = (-1)^t \times \begin{cases} 
B_n(1/3 - t) + B_n(2/3 - t), & 0 \leq t < 1/3; \\
B_n(4/3 - t) + B_n(2/3 - t), & 1/3 \leq t < 2/3; \\
B_n(4/3 - t) + B_n(5/3 - t), & 2/3 \leq t \leq 1. 
\end{cases} \]

Since \( G_n(t) \) is continuous for \( n \geq 2 \), \( G_n(1 - t) = (-1)^t G_n(t), 0 \leq t \leq 1 \), which proves the first identity. Further, we have \( F_n(t) = G_n(t) - G_n(0) \) and \( (-1)^t G_n(0) = G_n(0) \), since \( G_{2k-1}(0) = 0 \), so that

\[ F_n(1 - t) = G_n(1 - t) - G_n(0) = (-1)^t [G_n(t) - G_n(0)] = (-1)^t F_n(t), \]

which proves the second identity.

Note that the identities established in Lemma 1 are valid for \( n = 1 \) too except at the points \( 1/3 \) and \( 2/3 \) of discontinuity of \( F_1(t) = G_1(t) \).

**Lemma 2.** For \( k \geq 2 \) the function \( G_{2k-1}(t) \) has no zeros in the interval \((0, 1/2)\).
The sign of this function is determined by \( (-1)^t G_{2k-1}(t) > 0, 0 < t < 1/2 \).

**Proof.** For \( k = 2 \), \( G_3(t) \) is given by (2.9) and we have \( G_3(t) > 0 (0 < t < 1/2) \), so our assertion is true for \( k = 2 \). Now, assume that \( k \geq 3 \). Then \( 2k - 1 \geq 5 \) and \( G_{2k-1}(t) \) is continuous and twice differentiable. Using (1.12) we get

\[ G'_{2k-1}(t) = -(2k - 1)G_{2k-2}(t) \]

and

\[ G''_{2k-1}(t) = (2k - 1)(2k - 2)G_{2k-3}(t). \] (3.8)

We know that \( 0 \) and \( 1/2 \) are zeros of \( G_{2k-1}(t) \). Suppose that some \( \alpha \in (0, 1/2) \) is also a zero of \( G_{2k-1}(t) \). Then the derivative \( G'_{2k-1}(t) \) must have at least one zero \( \beta_1 \in (0, \alpha) \) and at least one zero \( \beta_2 \in (\alpha, 1/2) \). Therefore \( G''_{2k-1}(t) \) must have at least one zero inside \( (\beta_1, \beta_2) \). Thus, from the assumption that \( G_{2k-1}(t) \) has a zero inside \((0, 1/2)\), it follows from (3.8) that \( G_{2k-3}(t) \) also has a zero inside this interval, and so by induction \( G_3(t) \) has a zero on \((0, 1/2)\), which we have seen not to be the case. Hence \( G_{2k-1}(t) \) cannot have a zero on \((0, 1/2)\).

To determine the sign of \( G_{2k-1}(t) \), note that

\[ G_{2k-1}(1/3) = B_{2k-1}(0) + B_{2k-1}(1/3) = B_{2k-1}(1/3). \]

We have from [1, 23.1.14] that \( (-1)^t B_{2k-1}(t) > 0 (0 < t < 1/2) \), which implies

\[ (-1)^t G_{2k-1}(1/3) = (-1)^t B_{2k-1}(1/3) > 0. \]

Consequently \( (-1)^t G_{2k-1}(t) > 0 (0 < t < 1/2) \).
Corollary 1. For \( k \geq 2 \) the functions \((-1)^{k-1}F_{2k}(t)\) and \((-1)^{k-1}G_{2k}(t)\) are strictly increasing on \((0, 1/2)\) and strictly decreasing on \((1/2, 1)\). Consequently, 0 and 1 are the only zeros of \(F_{2k}(t)\) in \([0, 1]\) and

\[
\max_{t \in [0, 1]} |F_{2k}(t)| = 2(1 - 2^{-2k})(1 - 3^{1-2k})|B_{2k}|, \quad k \geq 2.
\]

Also \(\max_{t \in [0, 1]} |G_{2k}(t)| = (1 - 3^{1-2k})|B_{2k}|, k \geq 2\).

Proof. Using (1.12) we get \([(-1)^{k-1}F_{2k}(t)]^2 = \max_{t \in [0, 1]} |G_{2k}(t)|\] and \(-1)^kG_{2k-1}(t) > 0\) for \(0 < t < 1/2\) by Lemma 2. Thus \((-1)^kF_{2k}(t)\) and \((-1)^kG_{2k}(t)\) are strictly increasing on \((0, 1/2)\). Also by Lemma 1, \(F_{2k}(1-t) = F_{2k}(t)\) and \(G_{2k}(1-t) = G_{2k}(t)\) \((0 \leq t \leq 1)\), which implies that \((-1)^kF_{2k}(t)\) and \((-1)^kG_{2k}(t)\) are strictly decreasing on \((1/2, 1)\). Further, \(F_{2k}(0) = F_{2k}(1) = 0\), which implies that \(|F_{2k}(t)|\) achieves its maximum at \(t = 1/2\), that is,

\[
\max_{t \in [0, 1]} |F_{2k}(t)| = |F_{2k}(1/2)| = 2(1 - 2^{-2k})(1 - 3^{1-2k})|B_{2k}|.
\]

Also

\[
\max_{t \in [0, 1]} |G_{2k}(t)| = \max \{|G_{2k}(0)|, |G_{2k}(1/2)|\} = \max \{(1 - 3^{1-2k})|B_{2k}|, (1 - 2^{1-2k})(1 - 3^{1-2k})|B_{2k}|\} = (1 - 3^{1-2k})|B_{2k}|,
\]

which completes the proof.

Corollary 2. If \(k \geq 2\),

\[
\int_0^1 |F_{2k-1}(t)| \, dt = \int_0^1 |G_{2k-1}(t)| \, dt = \frac{2}{k}(1 - 2^{-2k})(1 - 3^{1-2k})|B_{2k}|.
\]

Also

\[
\int_0^1 |F_{2k}(t)| \, dt = |\tilde{B}_{2k}| = (1 - 3^{1-2k})|B_{2k}| \quad \text{and}
\]

\[
\int_0^1 |G_{2k}(t)| \, dt \leq 2|\tilde{B}_{2k}| = 2(1 - 3^{1-2k})|B_{2k}|.
\]

Proof. Using (1.12) we get

\[
G_m(t) = -mG_{m-1}(t), \quad m \geq 3.
\]

By (3.4) we have \(\int_0^1 |F_{2k-1}(t)| \, dt = \int_0^1 |G_{2k-1}(t)| \, dt\). By Lemmas 1 and 2 and (3.9) we get

\[
\int_0^1 |G_{2k-1}(t)| \, dt = 2 \int_0^{1/2} G_{2k-1}(t) \, dt = \frac{1}{k} |G_{2k}(1/2) - G_{2k}(0)|.
\]
The first assertion follows from (3.7) and (3.6).

From (3.5), (3.9) and the periodicity of $G_m$ for $m \geq 2$, we have

$$
\int_{0}^{1} F_{2k}(s) \, ds = (1 - 3^{1-2k}) B_{2k} = -B_{2k},
$$

(3.10)

by (3.3), which leads to the second assertion. Finally, we use (3.5) again and the triangle inequality to obtain

$$
\int_{0}^{1} |G_{2k}(t)| \, dt = \int_{0}^{1} |F_{2k}(t) + B_{2k}| \, dt \leq \int_{0}^{1} |F_{2k}(t)| \, dt + |B_{2k}| = 2|B_{2k}|,
$$

which proves the third assertion.

4. Two-point formula error estimates

In this section we use the two-point formulae of Euler type established in Theorem 1 to prove a number of inequalities for various classes of functions.

**THEOREM 2.** Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an $L$-Lipschitzian function on $[a, b]$ for some $n \geq 1$. Then

$$
\left| \int_{a}^{b} f(t) \, dt - M(a, b) + \tilde{T}_{n-1}(a, b) \right| \leq \frac{(b - a)^n}{2(n!)} \int_{0}^{1} |F_n(t)| \, dt \cdot L.
$$

(4.1)

Also

$$
\left| \int_{a}^{b} f(t) \, dt - M(a, b) + \tilde{T}_{n}(a, b) \right| \leq \frac{(b - a)^{n+1}}{2(n!)} \int_{0}^{1} |G_n(t)| \, dt \cdot L.
$$

(4.2)

**PROOF.** For any integrable function $\Phi : [a, b] \to \mathbb{R}$ we have

$$
\left| \int_{[a,b]} \Phi(t) \, d(f^{(n-1)}(t)) \right| \leq \int_{a}^{b} |\Phi(t)| \, dt \cdot L,
$$

(4.3)

since $f^{(n-1)}$ is $L$-Lipschitzian. Applying (4.3) with $\Phi(t) = F_n((t - a)/(b - a))$ gives

$$
\frac{(b - a)^n}{2(n!)} \int_{[a,b]} F_n \left( \frac{t - a}{b - a} \right) \, d(f^{(n-1)}(t)) \leq \frac{(b - a)^n}{2(n!)} \int_{a}^{b} |F_n \left( \frac{t - a}{b - a} \right)| \, dt \cdot L
$$

$$
= \frac{(b - a)^{n+1}}{2(n!)} \int_{0}^{1} |F_n(t)| \, dt \cdot L.
$$

Applying the above inequality, we get (4.1) from (2.4). Similarly, we can apply (4.3) with $\Phi(t) = G_n((t - a)/(b - a))$ and then use (2.3) to obtain (4.2).
COROLLARY 3. Let \( f : [a, b] \to \mathbb{R} \).
If \( f \) is \( L \)-Lipschitzian, then
\[
\left| \int_a^b f(t) \, dt - M(a, b) \right| \leq (5/36)(b-a)^2 \cdot L.
\]
If \( f' \) is \( L \)-Lipschitzian, then
\[
\left| \int_a^b f(t) \, dt - M(a, b) \right| \leq (1/36)(b-a)^3 \cdot L.
\]
If \( f'' \) is \( L \)-Lipschitzian, then
\[
\left| \int_a^b f(t) \, dt - M(a, b) \right| \leq \frac{13}{5184} (b-a)^4 \cdot L.
\]
If \( f''' \) is \( L \)-Lipschitzian, then
\[
\left| \int_a^b f(t) \, dt - M(a, b) \right| \leq \frac{13}{19440} (b-a)^5 \cdot L.
\]

PROOF. Using (2.6) and (2.7) we get \( \int_0^1 |F_1(t)| \, dt = 5/18 \) and \( \int_0^1 |F_2(t)| \, dt = 1/9 \), respectively. Therefore, using (2.5) and (2.1) and applying (4.1) with \( n = 1 \) and \( n = 2 \), we get the first and second inequalities, respectively. By Corollary 2, \( \int_0^1 |F_3(t)| \, dt = 13/432 \) and \( \int_0^1 |F_4(t)| \, dt = 13/405 \). The third inequality follows from (4.1) with \( n = 3 \) and (2.5), while the fourth follows from (4.1) with \( n = 4 \) and (2.5).

REMARK 2. For a function \( f \) which is \( L \)-Lipschitzian on \([a, b]\),
\[
\left| \int_a^b f(t) \, dt - \frac{b-a}{3} \left[ f(a) + f(b) + 2f \left( \frac{a+b}{2} \right) \right] \right| \leq \frac{5}{36} (b-a)^2 \cdot L
\]
(see [7] and [9]). This inequality is related to Simpson’s quadrature formula and gives an error estimate for an \( L \)-Lipschitzian function on \([a, b]\). This may be compared with the first inequality
\[
\left| \int_a^b f(t) \, dt - \frac{b-a}{2} \left[ f \left( \frac{2a+b}{3} \right) + f \left( \frac{a+2b}{3} \right) \right] \right| \leq \frac{5}{36} (b-a)^2 \cdot L
\]
in Corollary 3. We see that, for this class of function, we have the same error estimate for the two-point quadrature rule as for Simpson’s rule. However Simpson’s rule requires the evaluation of \( f \) at three points, while the two-point rule requires evaluation at two points only. Error estimates applying with the repeated use of these formulae for a finite interval consisting of \( n \) subintervals will also agree. In that context the Simpson scheme will involve evaluations at \( 2n + 1 \) points and our present procedure \( 2n \) points.

COROLLARY 4. Let \( f : [a, b] \to \mathbb{R} \) be such that \( f^{(n-1)} \) is \( L \)-Lipschitzian on \([a, b]\) for some \( n \geq 2 \). Set \( D_0(a, b) := 0 \) and for any integer \( r \) such that \( 1 \leq r \leq n/2 \) define
\[
D_r(a, b) := -\frac{1}{2} \sum_{i=1}^r \frac{(b-a)^{2i}}{(2i)!} (1 - 3^{1-2i}) B_{2i} \left[ f^{(2i-1)}(b) - f^{(2i-1)}(a) \right].
\]
If \( n = 2k - 1 \) \((k \geq 2)\), then
\[
\left| \int_a^b f(t) \, dt - M(a, b) + D_{k-1}(a, b) \right| \leq \frac{(b-a)^{2k}}{(2k)!} \left( 1 - 2^{-2k} \right) (1 - 3^{1-2k}) |B_{2k}| \cdot L.
\]
If \( n = 2k \) \((k \geq 2)\), then
\[
\left| \int_a^b f(t) \, dt - M(a, b) + D_{k-1}(a, b) \right| \leq \frac{(b-a)^{2k+1}}{2 \cdot (2k)!} (1 - 3^{1-2k}) |B_{2k}| \cdot L
\]
and
\[
\left| \int_a^b f(t) \, dt - M(a, b) + D_k(a, b) \right| \leq \frac{(b-a)^{2k+1}}{(2k)!} (1 - 3^{1-2k}) |B_{2k}| \cdot L.
\]

**Proof.** For \( n = 2k - 1 \) we have by \((4.5)\) that \( \tilde{T}_{n-1}(a, b) = D_{k-1}(a, b) \). Thus the first inequality follows from Corollary 2 and \((4.1)\). Moreover, for \( m \geq 2 \) we have that
\[
\tilde{T}_m(a, b) = \frac{1}{2} \sum_{k=1}^{[m/2]} \frac{(b-a)^{2k}}{(2k)!} B_{2k} \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right]
\]
\[
= -\frac{1}{2} \sum_{k=1}^{[m/2]} \frac{(b-a)^{2k}}{(2k)!} (1 - 3^{1-2k}) B_{2k} \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right], \tag{4.5}
\]
where \([x]\) denotes the greatest integer less than or equal to \( x \). Hence we have for \( n = 2k \) that \( \tilde{T}_{n-1}(a, b) = D_{k-1}(a, b) \) and \( \tilde{T}_n(a, b) = D_k(a, b) \). The second inequality follows from Corollary 2 and \((4.1)\) and the third from Corollary 2 and \((4.2)\).

**Remark 3.** Suppose that \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(n)} \) exists and is bounded on \([a, b] \), for some \( n \geq 1 \). In this case we have for all \( t, s \in [a, b] \) that
\[
\left| f^{(n-1)}(t) - f^{(n-1)}(s) \right| \leq \| f^{(n)} \|_\infty \cdot |t - s|,
\]
so that \( f^{(n-1)} \) is \( \| f^{(n)} \|_\infty \)-Lipschitzian on \([a, b]\). Therefore the inequalities established in Theorem 2 hold with \( L = \| f^{(n)} \|_\infty \). Consequently, under appropriate assumptions on \( f \), the inequalities from Corollary 3 hold with \( L = \| f' \|_\infty \), \( \| f'' \|_\infty \), \( \| f''' \|_\infty \) and \( \| f^{(n)} \|_\infty \), respectively. A similar observation can be made for the results of Corollary 4.

In the next theorem and subsequently we denote by \( V^h_a(f) \) the total variation of \( f \) on \([a, b]\).

**Theorem 3.** Let \( f : [a, b] \to \mathbb{R} \) be such that \( f^{(n-1)} \) is continuous and of bounded variation on \([a, b]\) for some \( n \geq 1 \). Then
\[
\left| \int_a^b f(t) \, dt - M(a, b) + \tilde{T}_{n-1}(a, b) \right| \leq \frac{(b-a)^n}{2(n!)} \max_{t \in [0,1]} |F'_n(t)| \cdot V^h_a(f^{(n-1)}) \tag{4.6}
\]
and
\[
\left| \int_a^b f(t) \, dt - M(a, b) + \tilde{T}_n(a, b) \right| \leq \frac{(b - a)^n}{2(n!)} \max_{t \in [0,1]} |G_n(t)| \cdot V^b_a \left( f^{(n)} \right). \tag{4.7}
\]

**PROOF.** If \( \Phi : [a, b] \to \mathbb{R} \) is bounded on \([a, b]\) and the Riemann-Stieltjes integral \( \int_{[a,b]} \Phi(t) \, df^{(n)}(t) \) exists, then
\[
\left| \int_{[a,b]} \Phi(t) \, df^{(n)}(t) \right| \leq \max_{t \in [a,b]} |\Phi(t)| \cdot V^b_a \left( f^{(n)} \right). \tag{4.8}
\]

We apply the estimate (4.8) to \( \Phi(t) = F_n((t - a)/(b - a)) \) to obtain
\[
\frac{(b - a)^n}{2(n!)} \int_{[a,b]} F_n \left( \frac{t-a}{b-a} \right) \, df^{(n)}(t) \leq \frac{(b - a)^n}{2(n!)} \max_{t \in [a,b]} |F_n(t)| \cdot V^b_a \left( f^{(n)} \right)
= \frac{(b-a)^{n+1}}{2(n!)} \max_{t \in [0,1]} |F_n(t)| \cdot V^b_a \left( f^{(n)} \right).
\]

We now use the above inequality and (2.4) to obtain (4.6). In the same way, we apply the estimate (4.8) to \( \Phi(t) = G_n((t - a)/(b - a)) \), and then use (2.3) to obtain (4.7).

**COROLLARY 5.** Let \( f : [a, b] \to \mathbb{R} \).

If \( f \) is continuous and of bounded variation on \([a, b]\), then
\[
\left| \int_a^b f(t) \, dt - M(a, b) \right| \leq \frac{b-a}{3} \cdot V^b_a(f).
\]

If \( f' \) is continuous and of bounded variation on \([a, b]\), then
\[
\left| \int_a^b f(t) \, dt - M(a, b) \right| \leq \frac{1}{18} (b-a)^2 \cdot V^b_a(f').
\]

If \( f'' \) is continuous and of bounded variation on \([a, b]\), then
\[
\left| \int_a^b f(t) \, dt - M(a, b) \right| - \frac{(b-a)^2}{36} \left[ f''(b) - f''(a) \right] \leq \frac{\sqrt{2}}{324} (b-a)^3 \cdot V^b_a(f'').
\]

If \( f''' \) is continuous and of bounded variation on \([a, b]\), then
\[
\left| \int_a^b f(t) \, dt - M(a, b) \right| - \frac{(b-a)^2}{36} \left[ f''(b) - f''(a) \right] \leq \frac{13}{10368} (b-a)^4 \cdot V^b_a(f''').
\]
**Proof.** From the explicit expressions (2.6), (2.8) and (2.9), we get
\[
\max_{t \in [0,1]} |F_1(t)| = -F_1(1/3) = 2/3, \quad \max_{t \in [0,1]} |F_2(t)| = F_2(1/3) = 2/9
\]
and
\[
\max_{t \in [0,1]} |F_3(t)| = F_3\left(\frac{1}{3\sqrt{2}}\right) = \frac{\sqrt{2}}{27},
\]
respectively. Therefore, using (2.5) and applying (4.6) with \(n = 1, 2, 3\), we get respectively the first, second and third inequalities. Further, by Corollary 1,
\[
\max_{t \in [0,1]} |F_4(t)| = 13/216.
\]
The fourth inequality follows from (4.6) with \(n = 4\) and (2.5).

**Remark 4.** It has been established in [8] (see also [9]) that
\[
\left| \int_a^b f(t) \, dt - \frac{b-a}{3} \left[ f(a) + f(b) + 2f\left(\frac{a+b}{2}\right) \right] \right| \leq \frac{b-a}{3} \cdot V^b_a(f).
\]
This inequality is related to Simpson’s quadrature formula and gives the error estimate for a function of bounded variation on \([a, b]\). This may be compared with the first inequality
\[
\left| \int_a^b f(t) \, dt - \frac{b-a}{2} \left[ f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right) \right] \right| \leq \frac{b-a}{3} \cdot V^b_a(f)
\]
in Corollary 5. The comparison in Remark 2 also applies here.

**Corollary 6.** Let \(f : [a, b] \to \mathbb{R}\) be such that \(f^{(n-1)}\) is continuous and of bounded variation on \([a, b]\) for some \(n \geq 2\). Define \(D_r(a, b) (r \geq 0)\) as in Corollary 4. If \(n = 2k - 1\) \((k \geq 2)\), then
\[
\left| \int_a^b f(t) \, dt - M(a, b) + D_{k-1}(a, b) \right| \leq \frac{b-a}{2} \frac{(2k-1)!}{(2k-1)!} \max_{t \in [0,1]} |F_{2k-1}(t)| \cdot V^b_a(f^{(2k-2)}).
\]
If \(n = 2k\) \((k \geq 2)\), then
\[
\left| \int_a^b f(t) \, dt - M(a, b) + D_{k-1}(a, b) \right| \leq \frac{(b-a)^{2k}}{(2k)!} (1 - 2^{-2k}) (1 - 3^{1-2k}) |B_{2k}| \cdot V^b_a(f^{(2k-1)})
\]
and
\[
\left| \int_a^b f(t) \, dt - M(a, b) + D_k(a, b) \right| \leq \frac{(b-a)^{2k}}{2} \frac{(2k)!}{(2k)!} (1 - 3^{1-2k}) |B_{2k}| \cdot V^b_a(f^{(2k-1)}).
\]
Two-point formulae 235

Theorem 3 and use the formulae established in Corollary 1.

Remark 5. Suppose that \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(n)} \in L_1[a, b] \) for some \( n \geq 1 \). In this case \( f^{(n-1)} \) is continuous and of bounded variation on \([a, b]\) and we have \( V_b^a(f^{(n-1)}) = \int_a^b |f^{(n)}(t)| \, dt = \|f^{(n)}\|_1 \). Therefore the inequalities established in Theorem 3 hold with \( \|f^{(n)}\|_1 \) in place of \( V_a^b(f^{(n-1)}) \). A similar observation can be made for the results of Corollaries 5 and 6.

Theorem 4. Suppose \((p, q)\) is a pair of conjugate exponents, which we may specify as \( 1 < p, q < \infty \) with \( p^{-1} + q^{-1} = 1 \) or \( p = \infty, q = 1 \), and let \( f : [a, b] \to \mathbb{R} \) be such that \( f^{(n)} \in L_p[a, b] \) for some \( n \geq 1 \). Then

\[
\left| \int_a^b f(t) \, dt - M(a, b) + \tilde{T}_{n-1}(a, b) \right| \leq K(n, p)(b - a)^{n+1/q} \cdot \|f^{(n)}\|_p, \tag{4.9}
\]

where \( K(n, p) = (1/2(n!))\left( \int_0^1 |F_n(t)|^q \, dt \right)^{1/q} \). Also

\[
\left| \int_a^b f(t) \, dt - M(a, b) + \tilde{T}_n(a, b) \right| \leq K^*(n, p)(b - a)^{n+1/q} \cdot \|f^{(n)}\|_p, \tag{4.10}
\]

where \( K^*(n, p) = (1/2(n!))\left( \int_0^1 |G_n(t)|^q \, dt \right)^{1/q} \).

Proof. By the H"older inequality, we have

\[
\frac{(b - a)^p}{2(n!)} \int_a^b F_n \left( \frac{t - a}{b - a} \right) f^{(n)}(t) \, dt \\
\leq \frac{(b - a)^p}{2(n!)} \left[ \int_a^b \left| F_n \left( \frac{t - a}{b - a} \right) \right|^q \, dt \right]^{1/q} \|f^{(n)}\|_p \\
= \frac{(b - a)^{n+1/q}}{2(n!)} \left[ \int_0^1 |F_n(t)|^q \, dt \right]^{1/q} \|f^{(n)}\|_p = K(n, p)(b - a)^{n+1/q} \|f^{(n)}\|_p.
\]

From this inequality, we get the estimate (4.6) from (2.4) and Remark 1. In the same way we get the estimate (4.10) from (2.3).

Remark 6. For \( p = \infty \) we have

\[
K(n, \infty) = \frac{1}{2(n!)} \int_0^1 |F_n(t)| \, dt \quad \text{and} \quad K^*(n, \infty) = \frac{1}{2(n!)} \int_0^1 |G_n(t)| \, dt.
\]
The results established in Theorem 4 for \( p = \infty \) coincide with those of Theorem 2 with \( L = \|f^{(\infty)}\|_{\infty} \). Moreover, by Remark 3 and Corollary 3, we have for \( n = 1, 2 \) that
\[
\left| \int_{a}^{b} f(t) \, dt - M(a, b) \right| \leq K(n, \infty)(b - a)^{n+1} \|f^{(n)}\|_{\infty},
\]
while for \( n = 3, 4 \) we have
\[
\left| \int_{a}^{b} f(t) \, dt - M(a, b) \right| \leq K(n, \infty)(b - a)^{n+1} \|f^{(n)}\|_{\infty},
\]
where \( K(1, \infty)=5/36, K(2, \infty)=1/36, K(3, \infty)=13/5184, K(4, \infty)=13/19440. \)

Further, by Remark 3 and Corollary 4, we have for \( k \geq 2 \) that
\[
\left| \int_{a}^{b} f(t) \, dt - M(a, b) + D_{k-1}(a, b) \right| \leq K(2k - 1, \infty)(b - a)^{2k} \|f^{(2k-1)}\|_{\infty},
\]
\[
\left| \int_{a}^{b} f(t) \, dt - M(a, b) + D_{k-1}(a, b) \right| \leq K(2k, \infty)(b - a)^{2k+1} \|f^{(2k)}\|_{\infty}
\]
and
\[
\left| \int_{a}^{b} f(t) \, dt - M(a, b) + D_{k}(a, b) \right| \leq K^*(2k, \infty)(b - a)^{2k+1} \|f^{(2k)}\|_{\infty},
\]
where
\[
K(2k - 1, \infty) = \frac{2 \left(1 - 2^{-2k}\right)(1 - 3^{1-2k})}{(2k)!} |B_{2k}|/2k,
\]
\[
K(2k, \infty) = \frac{1 - 3^{1-2k}}{2 \left(2k\right)!} |B_{2k}| \quad \text{and} \quad K^*(2k, \infty) \leq \frac{1 - 3^{1-2k}}{2 \left(2k\right)!} |B_{2k}|.
\]

**Remark 7.** For \( p = 1 \) define
\[
K(n, 1) := \frac{1}{2(n!)} \max_{t \in [0, 1]} |F_{a}(t)| \quad \text{and} \quad K^*(n, 1) := \frac{1}{2(n!)} \max_{t \in [0, 1]} |G_{a}(t)|.
\]

Then, using Remark 5 and Theorem 3, we can extend the results established in Theorem 4 to the pair \( p = 1, q = \infty \). Thus if we set \( 1/q = 0 \), then (4.9) and (4.10) hold for \( p = 1 \). Also, by Remark 5 and Corollary 5, we have for \( n = 1, 2 \) that
\[
\left| \int_{a}^{b} f(t) \, dt - M(a, b) \right| \leq K(n, 1)(b - a)^{n} \|f^{(n)}\|_{1},
\]
while for \( n = 3, 4 \) we have
\[
\left| \int_{a}^{b} f(t) \, dt - M(a, b) - \frac{(b - a)^2}{36} \left[ f'(b) - f'(a) \right] \right| \leq K(n, 1)(b - a)^{n} \|f^{(n)}\|_{1},
\]
where \( K(1, 1) = 1/3, K(2, 1) = 1/18, K(3, 1) = \sqrt{3}/324, K(4, 1) = 31/10368. \)

Further, by Remark 5 and Corollary 6, for \( k \geq 2 \) we have
\[
\left| \int_{a}^{b} f(t) \, dt - M(a, b) + D_{k+1}(a, b) \right| \leq K(2k - 1, 1)(b - a)^{2k-1} \|f^{(2k-1)}\|_{1},
\]

\[
\left| \int_{a}^{b} f(t) \, dt - M(a, b) + D_{k}(a, b) \right| \leq K(2k, 1)(b - a)^{2k} \|f^{(2k)}\|_{1}.
\]
Two-point formulae

\[ \int_a^b f(t) \, dt - M(a, b) + D_{2k-1}(a, b) \leq K(2k, 1)(b - a)^{2k} \| f^{(2k)} \|_1 \]

and

\[ \int_a^b f(t) \, dt - M(a, b) + D_k(a, b) \leq K^*(2k, 1)(b - a)^{2k} \| f^{(2k)} \|_1, \]

where

\[
K(2k - 1, 1) = \frac{1}{2[(2k - 1)!]} \max_{t \in [0,1]} |F_{2k-1}(t)|,
\]

\[
K(2k, 1) = \frac{(1 - 2^{-2k})(1 - 3^{1-2k})}{(2k)!} |B_{2k}| \quad \text{and} \quad K^*(2k, 1) = \frac{1 - 3^{1-2k}}{2[(2k)!]} |B_{2k}|.
\]

**Remark 8.** For \( 1 < p \leq \infty \) we can easily determine

\[
K(1, p) = \int_0^1 \left( \frac{2^{p+1} + 1}{3(q + 1)} \right)^{1/p} \, dt,
\]

so that for \( n = 1 \) Theorem 4 yields

\[
\int_a^b f(t) \, dt - \frac{b - a}{2} \left[ f \left( \frac{2a + b}{3} \right) + f \left( \frac{a + 2b}{3} \right) \right]
\leq \int_0^1 \left( \frac{2^{p+1} + 1}{3(q + 1)} \right)^{1/q} (b - a)^{1+1/q} \| f' \|_p.
\]

This may be compared with the similar inequality proved in [6] (see also [9]), related to Simpson’s rule

\[
\int_a^b f(t) \, dt - \frac{b - a}{3} \left[ f(a) + f(b) + 2f \left( \frac{a + b}{2} \right) \right]
\leq \int_0^1 \left( \frac{2^{p+1} + 1}{3(q + 1)} \right)^{1/q} (b - a)^{1+1/q} \| f' \|_p.
\]

The comparison in Remark 2 also applies here.

5. Quadrature formulae error estimates

Let us divide the interval \([a, b]\) into \( v \) subintervals of equal length \( h = (b - a)/v \). Assume that \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(n-1)} \) is continuous and of bounded variation on \([a, b]\), for some \( n \geq 1 \). We consider the repeated two-point quadrature formula

\[
\int_a^b f(t) \, dt = M_v(f) - \sigma_{n-1}(f) + \rho_n(f) \quad (5.1)
\]
and the repeated modified two-point quadrature formula

\[
\int_a^b f(t) \, dt = M_n(f) - \sigma_n(f) + \tilde{\rho}_n(f),
\]

(5.2)

where

\[
M_n(f) = \sum_{i=1}^v M(a + (i - 1)h, a + ih)
\]

\[
= \frac{h}{2} \sum_{i=1}^v [f(a + (i - 2/3)h) + f(a + (i - 1/3)h)]
\]

and \( \sigma_n(f) = \sum_{i=1}^v \tilde{T}_n(a + (i - 1)h, a + ih), m \geq 0. \)

Because of (2.5) we have

\[
\sigma_0(f) = \sigma_1(f) = 0,
\]

(5.3)

while for \( m \geq 2, \) we get using (4.5) that

\[
\sigma_m(f) = \frac{1}{2} \sum_{j=1}^{[m/2]} \frac{h^{2j}}{(2j)!} \tilde{B}_{2j} \left[ f^{(2j-1)}(a + ih) - f^{(2j-1)}(a + (i - 1)h) \right]
\]

\[
= \frac{1}{2} \sum_{j=1}^{[m/2]} \frac{h^{2j}}{(2j)!} \tilde{B}_{2j} \sum_{i=1}^v \left[ f^{(2j-1)}(a + ih) - f^{(2j-1)}(a + (i - 1)h) \right]
\]

\[
= -\frac{1}{2} \sum_{j=1}^{[m/2]} \frac{h^{2j}}{(2j)!} (1 - 3^{1-2j}) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right].
\]

(5.4)

The remainders \( \rho_n(f) \) and \( \tilde{\rho}_n(f) \) can be written as

\[
\rho_n(f) = \sum_{i=1}^v \rho_n(f; i), \quad \tilde{\rho}_n(f) = \sum_{i=1}^v \tilde{\rho}_n(f; i),
\]

(5.5)

where, for \( i = 1, \ldots, v, \)

\[
\rho_n(f; i) = \int_{a + (i - 1)h}^{a + ih} f(t) \, dt - M(a + (i - 1)h, a + ih) + \tilde{T}_{n-1}(a + (i - 1)h, a + ih)
\]

and

\[
\tilde{\rho}_n(f; i) = \int_{a + (i - 1)h}^{a + ih} f(t) \, dt - M(a + (i - 1)h, a + ih) + \tilde{T}_n(a + (i - 1)h, a + ih).
\]

We shall apply results from the preceding section to obtain some estimates for the remainders \( \rho_n(f) \) and \( \tilde{\rho}_n(f) \). Before doing this, note that for \( n = 2k - 1 \ (k \geq 2), \) we
have

\[ \sigma_{2k-2}(f) = \sigma_{2k-1}(f) = -\frac{1}{2} \sum_{j=1}^{k-1} \frac{h^{2j}}{(2j)!} (1 - 3^{1-2j}) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right]. \]

Thus \( \rho_{2k-1}(f) = \tilde{\rho}_{2k-1}(f) \), so that (5.1) and (5.2) coincide in this case. This shows that (5.2) is interesting only when \( n = 2k \) (\( k \geq 2 \)). In this case we have

\[ \tilde{\rho}_{2k}(f) = \rho_{2k}(f) + \sigma_{2k}(f) - \sigma_{2k-1}(f) \]

\[ = \rho_{2k}(f) - \frac{h^{2k}}{2(2k)!} (1 - 3^{1-2k}) B_{2k} \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right]. \]

In fact we have \( \tilde{\rho}_{2k-2}(f) = \rho_{2k}(f) \) \( (k \geq 2) \).

Therefore for \( k \geq 2 \) we can approximate \( f(t) \) by

\[ M_n(f) + \frac{1}{2} \sum_{j=1}^{k-1} \frac{h^{2j}}{(2j)!} (1 - 3^{1-2j}) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right], \]

using either (5.1) with \( n = 2k - 1 \) or (5.2) with \( n = 2k - 2 \). To obtain the error estimate for this approximation, if we apply (5.1), then we must assume that \( f^{(2k-2)} \) is continuous and of bounded variation on \([a, b]\). To do this via (5.2), it is enough to assume that \( f^{(2k-3)} \) is continuous and of bounded variation on \([a, b]\).

**Theorem 5.** Let \( f : [a, b] \to \mathbb{R} \) be such that \( f^{(n-1)} \) is \( L \)-Lipschitz on \([a, b]\) for some \( n \geq 1 \). For \( n = 1, 2, 3, 4 \) we have, respectively,

\[ \left| \int_a^b f(t) \, dt - M_n(f) \right| \leq \frac{5}{36} \nu h^3 L, \quad \left| \int_a^b f(t) \, dt - M_n(f) \right| \leq \frac{1}{36} \nu h^3 L, \]

\[ \left| \int_a^b f(t) \, dt - M_n(f) - \frac{h^2}{36} \left[ f'(b) - f'(a) \right] \right| \leq \frac{13}{5184} \nu h^5 L, \]

\[ \left| \int_a^b f(t) \, dt - M_n(f) - \frac{h^2}{36} \left[ f'(b) - f'(a) \right] \right| \leq \frac{13}{19440} \nu h^5 L. \]

If \( n = 2k - 1 \) \( (k \geq 2) \), then

\[ \left| \int_a^b f(t) \, dt - M_n(f) - \frac{1}{2} \sum_{j=1}^{k-1} \frac{h^{2j}}{(2j)!} (1 - 3^{1-2j}) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] \right| \]

\[ \leq \frac{\nu h^{2k}}{(2k)!} 2(1 - 2^{-2k})(1 - 3^{1-2k}) B_{2k} |L|. \]
If \( n = 2k \) \( (k \geq 2) \), then

\[
\left| \int_a^b f(t) \, dt - M_n(f) \right| \leq \frac{1}{2} \sum_{j=1}^{k-1} \frac{h^{2j}}{(2j)!} (1 - 3^{1-2j}) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] \\
\leq \frac{v h^{2k+1}}{2(2k)!} (1 - 3^{1-2k}) |B_{2k}| L
\]

and

\[
\left| \int_a^b f(t) \, dt - M_n(f) \right| \leq \frac{1}{2} \sum_{j=1}^{k-1} \frac{h^{2j}}{(2j)!} (1 - 3^{1-2j}) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] \\
\leq \frac{v h^{2k+1}}{(2k)!} (1 - 3^{1-2k}) |B_{2k}| L.
\]

**Proof.** Applying (4.1) and (4.2) we get for \( i = 1, \ldots, v \), respectively,

\[
|\rho_n(f; i)| \leq \frac{h^{n+1}}{2(n!)} \int_0^1 |F_n(t)| \, dt L \quad \text{and} \quad |\tilde{\rho}_n(f; i)| \leq \frac{h^{n+1}}{2(n!)} \int_0^1 |G_n(t)| \, dt L.
\]

Using the above estimates and the triangle inequality, we get from (5.5) that

\[
|\rho_n(f)| \leq \sum_{i=1}^v |\rho_n(f; i)| \leq \frac{v h^{n+1}}{2(n!)} \int_0^1 |F_n(t)| \, dt L
\]

and

\[
|\tilde{\rho}_n(f)| \leq \sum_{i=1}^v |\tilde{\rho}_n(f; i)| \leq \frac{v h^{n+1}}{2(n!)} \int_0^1 |G_n(t)| \, dt L.
\]

The rest of the argument, from (5.3) and (5.4), is as for Corollaries 3 and 4.

**Remark 9.** Instead of the assumption that \( f^{(n-1)} \) is \( L \)-Lipschitzian on \([a, b]\), we can use the stronger assumption that \( f^{(n)} \) exists and is bounded on \([a, b] \), for some \( n \geq 1 \). In this case Theorem 5 applies with \( L \) replaced by \( \|f^{(n)}\|_\infty \) (see Remark 3).

**Theorem 6.** Let \( f: [a, b] \to \mathbb{R} \) be such that \( f^{(n-1)} \) is continuous and of bounded variation on \([a, b]\) for some \( n \geq 1 \). For \( n = 1, 2, 3, 4 \) we have, respectively,

\[
\left| \int_a^b f(t) \, dt - M_n(f) \right| \leq \frac{1}{3} h V^b_a(f), \quad \left| \int_a^b f(t) \, dt - M_n(f) \right| \leq \frac{1}{18} h^2 V^b_a(f'), \\
\left| \int_a^b f(t) \, dt - M_n(f) - \frac{h^2}{36} [f'(b) - f'(a)] \right| \leq \frac{\sqrt{2}}{324} h^3 V^b_a(f''), \quad \left| \int_a^b f(t) \, dt - M_n(f) - \frac{h^2}{36} [f'(b) - f'(a)] \right| \leq \frac{13}{10368} h^3 V^b_a(f''').
\]
If \( n = 2k - 1 \) \((k \geq 2)\), then
\[
\left| \int_a^b f(t) \, dt - M_n(f) - \frac{1}{2} \sum_{j=1}^{k-1} \frac{h^{2j}}{(2j)!} (1 - 3^{1-2j}) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] \right|
\leq \frac{h^{2k-1}}{2!(2k-1)!} \max_{t \in [0,1]} |F_{2k-1}(t)| V^b_a(f^{(2k-2)}).
\]

If \( n = 2k \) \((k \geq 2)\), then
\[
\left| \int_a^b f(t) \, dt - M_n(f) - \frac{1}{2} \sum_{j=1}^{k} \frac{h^{2j}}{(2j)!} (1 - 3^{1-2j}) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] \right|
\leq \frac{h^{2k}}{(2k)!} (1 - 2^{-2k})(1 - 3^{1-2k}) |B_{2k}| V^b_a(f^{(2k-1)}).
\]

and
\[
\left| \int_a^b f(t) \, dt - M_n(f) - \frac{1}{2} \sum_{j=1}^{k} \frac{h^{2j}}{(2j)!} (1 - 3^{1-2j}) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] \right|
\leq \frac{h^{2k}}{2!(2k)!} (1 - 3^{1-2k}) |B_{2k}| V^b_a(f^{(2k-1)}).
\]

**Proof.** Applying (4.6) and (4.7) we get for \( i = 1, \ldots, v \) respectively that
\[
|\rho_n(f;i)| \leq \frac{h^n}{2(n!)} \max_{t \in [0,1]} |F_n(t)| V^{a+h}_{a+(i-1)h}(f^{(n-1)}),
\]
and
\[
|\tilde{\rho}_n(f;i)| \leq \frac{h^n}{2(n!)} \max_{t \in [0,1]} |G_n(t)| V^{a+h}_{a+(i-1)h}(f^{(n-1)}).
\]

Using the above estimates and the triangle inequality, we get from (5.5) that
\[
|\rho_n(f)| \leq \sum_{i=1}^{v} |\rho_n(f;i)| \leq \frac{h^n}{2(n!)} \max_{t \in [0,1]} |F_n(t)| \sum_{i=1}^{v} V^{a+h}_{a+(i-1)h}(f^{(n-1)})
\]
\[
= \frac{h^n}{2(n!)} \max_{t \in [0,1]} |F_n(t)| V^b_a(f^{(n-1)})
\]
and similarly \( |\tilde{\rho}_n(f)| \leq (h^n/2(n!)) \max_{t \in [0,1]} |G_n(t)| V^b_a(f^{(n-1)}) \). We now use (5.3) and (5.4) and argue as in Corollaries 5 and 6.

**Remark 10.** If \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(n)} \in L_1[a, b] \) for some \( n \geq 1 \), then \( f^{(n-1)} \) is continuous and of bounded variation on \([a, b]\) and \( V^b_a(f^{(n-1)}) = \| f^{(n)} \|_1 \). Therefore Theorem 6 applies with \( \| f^{(n)} \|_1 \) in place of \( V^b_a(f^{(n-1)}) \) (see Remark 5).
THEOREM 7. Assume \((p, q)\) is a pair of conjugate exponents. Let \(f : [a, b] \to \mathbb{R}\) be such that \(f^{(n)} \in L_p[a, b]\) for some \(n \geq 1\). Then \(|\rho_n(f)| \leq vK(n, p)h^{n+1/q}||f^{(i)}||_p\) and \(|\tilde{\rho}_n(f)| \leq vK^*(n, p)h^{n+1/q}||f^{(i)}||_p\), where \(K(n, p)\) and \(K^*(n, p)\) are defined as in Theorem 4.

PROOF. For \(i = 1, \ldots, v\) let \(g_i(t) = f^{(i)}(t), t \in [a + (i - 1)h, a + ih]\). Then \(\|g_i\|_p \leq \|f^{(i)}\|_p\), where the norm \(\|g_i\|_p\) is taken over the interval \([a + (i - 1)h, a + ih]\), while the norm \(\|f^{(i)}\|_p\) is taken over the interval \([a, b]\). Applying (4.9) and (4.10) and using the above inequality, we get for \(i = 1, \ldots, v\) that

\[
|\rho_n(f; i)| \leq K(n, p)h^{n+1/q}\|g_i\|_p \leq K(n, p)h^{n+1/q}||f^{(i)}||_p
\]

and

\[
|\tilde{\rho}_n(f; i)| \leq K^*(n, p)h^{n+1/q}\|g_i\|_p \leq K^*(n, p)h^{n+1/q}||f^{(i)}||_p.
\]

The result follows from (5.5) by the triangle inequality.

In the following discussion we assume that \(f : [a, b] \to \mathbb{R}\) has a continuous derivative of order \(n\), for some \(n \geq 1\). In this case we can use (2.4) and the second formula from Remark 1 to obtain, for \(i = 1, \ldots, v\), that

\[
\rho_n(f; i) = \frac{h^n}{2(n!)} \int_{a+(i-1)h}^{a+ih} F_n \left( \frac{t-a-(i-1)h}{h} \right) f^{(i)}(t) \, dt
\]

\[
= \frac{h^{n+1}}{2(n!)} \int_0^1 F_n(s) f^{(i)}(a+(i-1)h+hs) \, ds.
\]

Therefore we get by (5.5) that

\[
\rho_n(f) = \frac{h^{n+1}}{2(n!)} \int_0^1 F_n(s) \Phi_n(s) \, ds,
\]

where

\[
\Phi_n(s) = \sum_{i=1}^v f^{(i)}(a+(i-1)h+hs), \quad 0 \leq s \leq 1.
\]

Similarly, we get \(\tilde{\rho}_n(f) = (h^{n+1}/2(n!)) \int_0^1 G_n(s) \Phi_n(s) \, ds\). Obviously, \(\Phi_n(s)\) is continuous on \([0, 1]\) and

\[
\int_0^1 \Phi_n(s) \, ds = h^{-1} \sum_{i=1}^v \left[ f^{(a-1)}(a + ih) - f^{(a-1)}(a + (i-1)h) \right]
\]

\[
= h^{-1} \left[ f^{(a-1)}(b) - f^{(a-1)}(a) \right].
\]
From the discussion at the beginning of this section, the most interesting case is the repeated two-point quadrature formula of Euler type (5.1) for \( n = 2k \) \((k \geq 2)\), which can be rewritten as

\[
\int_a^b f(t) \, dt = M_n(f) + \frac{h}{2} \sum_{j=1}^{k-1} \frac{h^{2j}}{(2j)!} (1 - 3^{1-2j})B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] + \rho_{2k}(f). \tag{5.9}
\]

The empty sum for \( k = 1 \) is taken as zero.

**Theorem 8.** If \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(2k)} \) is continuous on \([a, b]\), for some \( k \geq 1 \), then there exists a point \( \eta \in [a, b] \) such that

\[
\rho_{2k}(f) = \frac{h^{2k+1}}{2(2k)!} (1 - 3^{1-2k})B_{2k} f^{(2k)}(\eta). \tag{5.10}
\]

**Proof.** Using (5.6), we can rewrite \( \rho_{2k}(f) \) as

\[
\rho_{2k}(f) = (-1)^{k-1} \frac{h^{2k+1}}{2(2k)!} J_k, \tag{5.11}
\]

where

\[
J_k = \int_0^1 (-1)^{k-1} F_{2k}(s) \Phi_{2k}(s) \, ds. \tag{5.12}
\]

If \( m = \min_{t \in [a,b]} f^{(2k+1)}(t) \), \( M = \max_{t \in [a,b]} f^{(2k+1)}(t) \), then we get from (5.7) that \( vm \leq \Phi_{2k}(s) \leq vM, 0 \leq s \leq 1 \). On the other hand, (2.8) and Corollary 1 give

\[
(1)^{k-1} F_{2k}(s) \geq 0, \quad 0 \leq s \leq 1,
\]

which implies \( vm \int_0^1 (-1)^{k-1} F_{2k}(s) \, ds \leq J_k \leq vM \int_0^1 (-1)^{k-1} F_{2k}(s) \, ds \). Using (3.10) we have \( vm(-1)^k \tilde{B}_{2k} \leq J_k \leq vM(-1)^k \tilde{B}_{2k} \). By the continuity of \( f^{(2k)}(s) \) on \([a, b]\), it follows that there must exist a point \( \eta \in [a, b] \) such that \( J_k = v(-1)^k \tilde{B}_{2k} f^{(2k)}(\eta) \). Combining this with (5.11) and (3.3) gives (5.10).

**Remark 11.** The repeated two-point quadrature formula of Euler type (5.9) is a generalisation of the two-point formula (1.2). Namely, from (5.10) for \( k = 1 \) and \( v = 1 \) we get \( \rho_2(f) = \left( (b-a)^3/36 \right) f''(\eta) \) and (5.9) reduces to (1.2).

**Theorem 9.** If \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(2k)} \) is continuous on \([a, b]\), for some \( k \geq 1 \), and does not change sign on \([a, b]\), then there exists a point \( \theta \in [0, 1] \) such that

\[
\rho_{2k}(f) = \frac{h^{2k}}{(2k)!} (1 - 2^{1-2k})(1 - 3^{1-2k})B_{2k} \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right]. \tag{5.13}
\]
PROOF. Suppose that \( f^{(2k)}(t) \geq 0, a \leq t \leq b \). Then from (5.7) we get \( \Phi_{2k}(s) \geq 0 \), \( 0 \leq s \leq 1 \). It follows from Corollary 1 that \( 0 \leq (-1)^{k-1} F_{2k}(s) \leq (-1)^{k-1} F_{2k}(1/2), 0 \leq s \leq 1 \). Therefore if \( J_k \) is given by (5.12), \( 0 \leq J_k \leq (-1)^{k-1} F_{2k}(1/2) \int_0^1 \Phi_{2k}(s) \, ds \).

Using (5.8), we get

\[
0 \leq J_k \leq (-1)^{k-1} 2(1 - 2^{-2k})(1 - 3^{1-2k}) B_{2k} h^{-1} \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right],
\]

which means that there must exist a point \( \theta \in [0, 1] \) such that

\[
J_k = \theta (-1)^{k-1} 2(1 - 2^{-2k})(1 - 3^{1-2k}) B_{2k} h^{-1} \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right].
\]

Combining this with (5.11) gives (5.13). When \( f^{(2k)}(t) \leq 0 (a \leq t \leq b) \) the argument is the same, in that case we get

\[
(-1)^{k-1} 2(1 - 2^{-2k})(1 - 3^{1-2k}) B_{2k} h^{-1} \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] \leq J_k \leq 0.
\]

REMARK 12. If we approximate \( \int_a^b f(t) \, dt \) by

\[
I_{2k}(f) = M_2(f) + \frac{1}{2} \sum_{j=1}^{k-1} \frac{h^{2j}}{(2j)!} \left( 1 - 3^{1-2j} \right) B_{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right],
\]

then the next approximation will be \( I_{2k+2}(f) \). The difference \( \Delta_{2k}(f) := I_{2k+2}(f) - I_{2k}(f) \) is equal to the last term in the sum in \( I_{2k+2}(f) \), that is,

\[
\Delta_{2k}(f) = \frac{h^{2k}}{2[(2k + 2)!]} \left( 1 - 3^{1-2k} \right) B_{2k} \left[ f^{(2k-1)}(b) - f^{(2k-1)}(a) \right]. \tag{5.14}
\]

We see that, under the assumptions of Theorem 9, \( \rho_{2k}(f) \) and \( \Delta_{2k}(f) \) are of the same sign. Moreover, we have \( \rho_{2k}(f) = 2\theta(1 - 2^{-2k}) \Delta_{2k}(f) \), which yields the simple estimate \( |\rho_{2k}(f)| \leq 2|\Delta_{2k}(f)| \) for the remainder \( \rho_{2k}(f) \).

THEOREM 10. Suppose that \( f : [a, b] \to \mathbb{R} \) is such that \( f^{(2k+2)} \) is continuous on \([a, b]\), for some \( k \geq 1 \). If for each \( x \in [a, b] \), \( f^{(2k)}(x) \) and \( f^{(2k+2)}(x) \) are either both nonnegative or both nonpositive, then the remainder \( \rho_{2k}(f) \) has the same sign as the first neglected term \( \Delta_{2k}(f) \) given by (5.14). Moreover, we have the estimate \( |\rho_{2k}(f)| \leq |\Delta_{2k}(f)| \).

PROOF. We have \( \Delta_{2k}(f) + \rho_{2k+2}(f) = \rho_{2k}(f) \), that is,

\[
\Delta_{2k}(f) = -\rho_{2k+2}(f) + \rho_{2k}(f). \tag{5.15}
\]

By (5.6)

\[
-\rho_{2k+2}(f) = \frac{h^{2k+3}}{2((2k + 2)!)} \int_0^1 [-F_{2k+2}(s)] \Phi_{2k+2}(s) \, ds.
\]
and
\[ \rho_{2k}(f) = \frac{h^{2k+1}}{2((2k)!)^2} \int_0^1 F_{2k+2}(s) \Phi_{2k}(s) \, ds. \]

Under the assumptions made on \( f \) we see that for all \( s \in [0, 1] \), \( \Phi_{2k}(s) \) and \( \Phi_{2k+2}(s) \) are either both nonnegative or both nonpositive. Also, from (2.8) and Corollary 1 it follows that for all \( s \in [0, 1] \), \((-1)^{k-1}[F_{2k+2}(s)] \geq 0 \) and \((-1)^{k-1} F_{2k}(s) \geq 0 \).

We conclude that \(-\rho_{2k+2}(f)\) and \(\rho_{2k}(f)\) have the same sign. Because of (5.15), \( \Delta_{2k}(f) \) must therefore have the same sign as \(-\rho_{2k+2}(f)\) and \(\rho_{2k}(f)\). Moreover, it follows that \(| -\rho_{2k+2}(f) | \leq | \Delta_{2k}(f) | \) and \(| \rho_{2k}(f) | \leq | \Delta_{2k}(f) | \).

References