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DISCRETE EQUATIONS CORRESPONDING TO FOURTH-ORDER DIFFERENTIAL EQUATIONS OF THE P_2 AND K_2 HIERARCHIES

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Abstract

Using the Bäcklund transformations for the solutions of fourth-order differential equations of the P_2 and K_2 hierarchies, corresponding discrete equations are found.

1. Introduction

In recent years interest in integrable mappings [27], discrete equations and especially discrete Painlevé equations [9] has grown considerably. The latter appear in the theory of orthogonal polynomials and some physical applications. For example they arise in two-dimensional quantum gravity [13, 14, 24, 26]. Discrete Painlevé equations have many properties in common with those of the usual Painlevé equations. They can be presented in the form of Lax pairs, they also have Bäcklund transformations and can be written in bilinear form. Some of them have special solutions for specific parameter values.

Discrete equations are conceptually identical to recursion formulas that connect solutions of differential equations for different parameter values. From mathematical physics we know the recursion formulas

$$J_{\nu+1}(x) + J_{\nu-1}(x) = \frac{2\nu}{x} J_\nu(x),$$
$$(n+1)P_{n+1}(x) - (2n+1)x P_n(x) + n P_{n-1}(x) = 0,$$
$$H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0,$$

where $J_\nu(x)$ is the Bessel function, $P_n(x)$ is the Legendre polynomial and $H_n(x)$ is the Chebyshev-Hermite polynomial. Certainly all these recursion formulas can be considered as discrete equations for $J_\nu(x)$, $P_n(x)$ and $H_n(x)$.

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It is known [7, 10, 25] that discrete Painlevé equations can be obtained using Bäcklund transformations for solutions of corresponding differential equations. This approach was realized in [4] where discrete equations for the PII, PIII and PIV equations were found.

Let us demonstrate this method to obtain the discrete equation corresponding to the second Painlevé equation [7]

$$y_{zz} = 2y^3 + zy + \alpha. \tag{1.1}$$

The Bäcklund transformations for the solutions of (1.1) are known. They have the form [1, 2, 23]

$$y(z,\alpha+1) = -y(z,\alpha) - \frac{2\alpha+1}{2y^2(z,\alpha)+2y_z(z,\alpha)+z}, \quad \alpha \neq -1/2, \tag{1.2}$$

$$y(z,\alpha-1) = -y(z,\alpha) - \frac{2\alpha-1}{2y^2(z,\alpha)-2y_z(z,\alpha)+z}, \quad \alpha \neq 1/2. \tag{1.3}$$

Substituting to remove the derivative $y_z(z,\alpha)$ from (1.2) and (1.3) and rearranging we have

$$\frac{2\alpha+1}{y(z,\alpha+1)+y(z,\alpha)} + \frac{2\alpha-1}{y(z,\alpha)+y(z,\alpha-1)} + 4y^2(z,\alpha) + 2z = 0. \tag{1.4}$$

Substituting $y(z,\alpha) = x_n$, $\alpha = \alpha_n = n + k - 1/2$, where k is an arbitrary parameter, n is a number and x_n is a variable into (1.4) we get a discrete equation of the form

$$\frac{n+k}{x_{n+1}+x_n} + \frac{n-1+k}{x_n+x_{n-1}} + 2x_n^2 + z = 0 \tag{1.5}$$

where z is the parameter of (1.5).

This approach for constructing higher order discrete equations corresponding to the second Painlevé hierarchy was formulated in [4]. Later we shall present the solution of this problem for two fourth-order differential equations.

Recently hierarchies of nonlinear differential equations with solutions having properties similar to those of the solutions of the Painlevé equations were introduced. This enables hierarchies of the first and second Painlevé equations to be presented in the form [16, 19, 20]

$$L^{n+1}[y] = z, \quad (n = 1, 2, \ldots), \tag{1.6}$$

$$\left[\frac{d}{dz} + 2y\right] L^n\left[y_z - y^2\right] - zy - \alpha = 0, \tag{1.7}$$

where the operator L^n is determined by the recursion formula [28, 29]

$$\frac{d}{dz}L^{n+1} = L^n_{zzz} + 4yL^n_z + 2y_zL^n, \quad L^0[y] = 1, \quad L^1[y] = y. \tag{1.8}$$

The first Painlevé hierarchy (1.6) was introduced in [16], but the second Painlevé hierarchy (1.7) was considered earlier [1, 6, 11, 12].

Two new hierarchies of nonlinear differential equations were presented in [19, 20]. They take the form

$$H_{n+1}[y] = z, \quad (n = 1, 2, \dots), \tag{1.9}$$

and

$$\left(\frac{d}{dz} + 2y\right) H_n\left[y_z - y^2\right] - zy - \beta = 0 \tag{1.10}$$

where the operator $H_n[y]$ is determined by the recursion formulas [3, 5, 22]

$$H_{n+2} = J[y]\theta[y]H_n, \quad H_0[y] = 1, \quad H_1[y] = y_{zz} + 4y^2$$
$$J = D^3 + 3(yD + Dy) + 2(D^2yD^{-1} + D^{-1}yD^2) + 8(y^2D^{-1} + D^{-1}y^2)$$
$$D = d/dz, \quad D^{-1} = \int dz, \quad \theta = D^3 + 2yD + y_z.$$

Taking into account the recent 150-th anniversary of Sophie Kovalevski's birth and Martin Kruskal's 75-th birthday let us call hierarchies (1.9) and (1.10) the K_1 and K_2 hierarchies respectively.

It was shown in [5, 17, 22] that fourth-order differential equations obtained from (1.6), (1.7), (1.9) and (1.10) at $n = 2$ have solutions in the form of transcendental functions with respect to constants of integration [17, 18].

Using Bäcklund transformations for the solutions of the fourth-order differential equations of the P_2 and K_2 hierarchies, we shall first present discrete equations corresponding to (1.7) and (1.10).

2. Discrete equations corresponding to the fourth-order differential equation of the P_2 hierarchy

From the P_2 hierarchy (1.7) for $n = 1$ we have the second Painlevé equation (1.1), but in the case that $n = 2$ we get the fourth-order differential equation in the form

$$y_{zzzz} - 10y^2y_{zz} - 10yy_z^2 + 6y^5 - zy - \alpha = 0. \tag{2.1}$$

In this section we are going to find the discrete equations corresponding to (2.1). Bäcklund transformations for solutions of (2.1) have the form [6, 8, 11, 12, 17]

$$y(z, \alpha+1) = -y(z, \alpha) - \frac{1+2\alpha}{2y_{zzz} + 4yy_{zz} - 2y_z^2 - 12y^2y_z - 6y^4 + z}, \quad \alpha \neq -\frac{1}{2}, \tag{2.2}$$

$$y(z, \alpha-1) = -y(z, \alpha) - \frac{1-2\alpha}{2y_{zzz} - 4yy_{zz} + 2y_z^2 - 12y^2y_z + 6y^4 - z}, \quad \alpha \neq \frac{1}{2}, \tag{2.3}$$

where $y = y(z, \alpha)$.

Relations (2.2) and (2.3) can be written in the form

$$2y_{zzz} + 4yy_{zz} - 2y_z^2 - 12y^2y_z - 6y^4 + z = -\frac{2\alpha + 1}{y(z, \alpha) + y(z, \alpha + 1)}, \tag{2.4}$$

$$2y_{zzz} - 4yy_{zz} + 2y_z^2 - 12y^2y_z + 6y^4 - z = \frac{2\alpha - 1}{y(z, \alpha) + y(z, \alpha - 1)}. \tag{2.5}$$

For solutions of (2.1) we are going to prove the following lemma.

LEMMA 2.1. *Let $y(z, \alpha - 1)$ and $y(z, \alpha)$ be solutions of* (2.1), *then we have*

$$y_z(z, \alpha) + y_z(z, \alpha - 1) = y^2(z, \alpha) - y^2(z, \alpha - 1). \tag{2.6}$$

PROOF. Replace α by $\alpha - 1$ in (2.4). Equate the left parts of the equation found with (2.5). We then get (2.6) after transformations.

REMARK 2.1. Relation (2.6) is valid for all solutions of the P_2 hierarchy. This statement can be proved strictly by analogy with Lemma 2.1 considering the Bäcklund transformations for the P_2 hierarchy and properties of the Lenard operator (1.8).

Let us denote

$$\begin{aligned} &y(z, \alpha - 2) = p, \quad y(z, \alpha - 1) = h, \quad y(z, \alpha) = f, \\ &y(z, \alpha + 1) = g, \quad y(z, \alpha + 2) = r. \end{aligned} \tag{2.7}$$

The sum and the difference of (2.4) and (2.5) can then be written in the form

$$\begin{aligned} 4f_{zzz} - 24f^2f_z &= \frac{2\alpha - 1}{f + h} - \frac{2\alpha + 1}{f + g}, \\ 8ff_{zz} - 4f_z^2 - 12f^4 + 2z &= -\frac{2\alpha - 1}{f + h} - \frac{2\alpha + 1}{f + g}. \end{aligned} \tag{2.8}$$

Taking into account (2.7) and Lemma 2.1, we have the following corollary.

COROLLARY 2.1. *Let p, h, f, g, r be solutions of the P_2 hierarchy determined by* (2.7). *The following equalities then hold*:

$$g_z = g^2 - f^2 - f_z, \tag{2.9}$$

$$g_{zz} = -f_{zz} - 2ff_z + 2g^3 - 2gf^2 - 2gf_z, \tag{2.10}$$

$$h_z = f^2 - h^2 - f_z, \tag{2.11}$$

$$h_{zz} = -f_{zz} + 2ff_z - 2hf^2 + 2h^3 + 2hf_z. \tag{2.12}$$

Replacing α by $\alpha+1$ and $\alpha-1$ in (2.8), we get solutions for g and h respectively:

$$8gg_{zz}-4g_z^2-12g^4+2z+\frac{2\alpha+1}{f+g}+\frac{2\alpha+3}{g+r}=0, \tag{2.13}$$

$$8hh_{zz}-4h_z^2-12h^4+2z+\frac{2\alpha-3}{h+p}+\frac{2\alpha-1}{f+h}=0. \tag{2.14}$$

Substitute to remove g_z and g_{zz} from (2.13) by means of (2.9) and (2.10). Next remove derivatives h_z and h_{zz} from (2.14) using (2.11) and (2.12). Then excluding f_{zz} by means of (2.8) from the relations obtained and multiplying by $f/(f+g)$ and $f/(f+h)$ respectively, we finally get the following equalities:

$$\begin{aligned}&-4f_z^2-8f^2f_z-8gff_z-4f^4-8f^3g+2z+\frac{2\alpha-1}{f+h}\\&\qquad+\frac{2\alpha+1}{f+g}-\frac{(2\alpha-1)f}{(f+h)(f+g)}+\frac{(2\alpha+3)f}{(f+g)(g+r)}=0,\end{aligned} \tag{2.15}$$

$$\begin{aligned}&-4f_z^2+8f^2f_z+8hff_z-4f^4-8f^2h+2z+\frac{2\alpha-1}{f+h}\\&\qquad+\frac{2\alpha+1}{f+g}-\frac{(2\alpha+1)f}{(f+h)(f+g)}+\frac{(2\alpha-3)f}{(f+h)(h+p)}=0.\end{aligned} \tag{2.16}$$

Subtracting (2.15) from (2.16), we find an expression for f_z of the form

$$\begin{aligned}f_z=\frac{1}{8(2f+g+h)}\Bigg[&8f^2(h-g)+\frac{2}{(f+g)(f+h)}\\&+\frac{2\alpha+3}{(f+g)(g+r)}-\frac{2\alpha-3}{(f+h)(h+p)}\Bigg].\end{aligned} \tag{2.17}$$

Substituting (2.17) into (2.15), we obtain a recursion formula for solutions of (2.1) for different values of parameter α:

$$\begin{aligned}&\Bigg[f(f+g)+\frac{1}{2f+g+h}\Bigg(f^2(h-g)+\frac{1}{4(f+h)(f+g)}-\frac{2\alpha-3}{8(f+h)(h+p)}\\&\quad+\frac{2\alpha+3}{8(f+g)(g+r)}\Bigg)\Bigg]^2-g^2f^2-\frac{z}{2}-\frac{2\alpha+1}{4(f+g)}-\frac{g(2\alpha-1)}{4(f+g)(f+h)}\\&\quad-\frac{f(2\alpha+3)}{4(f+g)(g+r)}=0.\end{aligned} \tag{2.18}$$

We note that (2.18) is not valid for $\alpha=-3/2,-1/2,1/2,3/2$.

We can find a similar equation by substituting (2.17) into (2.16). However, calculating r at given solutions p, h, f and g, we get two roots and one of them does not satisfy (2.1).

The unique solution of (2.1) can be found using (2.9) and (2.17). The recursion formula looking for six solutions of (2.1) takes the form

$$g^2 - f^2 - \left[f^2(h-g) + \frac{1}{4(f+g)(f+h)} + \frac{2\alpha+3}{8(f+g)(g+r)} - \frac{2\alpha-3}{8(f+h)(h+p)} \right] \frac{1}{(2f+g+h)} - \frac{1}{(2g+r+f)} \left[g^2(f-r) + \frac{1}{4(f+g)(g+r)} + \frac{2\alpha+5}{8(g+r)(r+d)} - \frac{2\alpha-1}{8(f+g)(f+h)} \right] = 0, \qquad (2.19)$$

where $d = y(z, \alpha+3)$. Equation (2.19) is not valid for $\alpha = -5/2, -3/2, -1/2, 1/2, 3/2$. This equation allows us to find solutions $d = y(z, \alpha+3)$ or $p = y(z, \alpha-2)$ for the given five solutions of (2.1).

Assuming $x_n = y(z, \alpha_n)$, $\alpha_{n+1} = \alpha_n + 1$, we have a discrete equation from (2.19) in the form

$$x_{n+1}^2 - x_n^2 - \frac{1}{2x_n + x_{n+1} + x_{n-1}} \left(x_n^2(x_{n-1} - x_{n+1}) + \frac{1}{4(x_n + x_{n+1})(x_n + x_{n-1})} + \frac{2\alpha_n + 3}{8(x_n + x_{n+1})(x_{n+1} + x_{n+2})} - \frac{2\alpha_n - 3}{8(x_n + x_{n-1})(x_{n-1} + x_{n-2})} \right) - \frac{1}{2x_{n+1} + x_{n+2} + x_n} \left(x_{n+1}^2(x_n - x_{n+2}) + \frac{1}{4(x_n + x_{n+1})(x_{n+1} + x_{n+2})} + \frac{2\alpha_n + 5}{8(x_{n+1} + x_{n+2})(x_{n+2} + x_{n+3})} - \frac{2\alpha_n - 1}{8(x_n + x_{n+1})(x_n + x_{n-1})} \right) = 0. \qquad (2.20)$$

One can see that this equation is not valid for $\alpha_n = -5/2, -3/2, -1/2, 1/2, 3/2$. Equation (2.20) can be used to find the unique solution x_{n+3} of (2.1) for given x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}.

3. Recursion formulas corresponding to the fourth-order differential equation of the K_2 hierarchy

Assuming $n = 1$ in (1.10), we get a fourth order differential equation in the form [20]

$$y_{zzzz} + 5y_z y_{zz} - 5yy_z^2 - 5y^2 y_{zz} + y^5 - zy - \beta = 0. \qquad (3.1)$$

Later in this section we shall suggest the discrete equations which correspond to (3.1).

Solutions of (3.1) are essentially transcendental functions with respect to constants of integration and have properties like those for the solutions of the Painlevé equations [8, 20]. There are Bäcklund transformations for solutions of (3.1). They allow us to obtain solutions of (3.1) for a given solution $y(z, \beta)$. These transformations have the form [15]

$$y(z, 2-\beta) = y(z, \beta) + \frac{2\beta - 2}{z - y_{zzz} + yy_{zz} - 3y_z^2 + 4y^2 y_z - y^4}, \qquad \beta \neq 1, \tag{3.2}$$

$$y(z, -1-\beta) = y(z, \beta) + \frac{2\beta + 1}{z + 2y_{zzz} + 4yy_{zz} + 3y_z^2 - 2y^2 y_z - y^4}, \quad \beta \neq -1/2, \tag{3.3}$$

where we denote $y = y(z, \beta)$ on the right-hand sides of (3.2) and (3.3).

LEMMA 3.1. *Let $y(z, \beta)$, $y(z, 2-\beta)$ and $y(z, -1-\beta)$ be solutions of* (3.1). *The equalities*

$$y_z(z, \beta) - y_z(z, 2-\beta) = \frac{1}{2} y^2(z, \beta) - \frac{1}{2} y^2(z, 2-\beta), \tag{3.4}$$

$$y_z(z, \beta) - y_z(z, -1-\beta) = -y^2(z, \beta) + y^2(z, -1-\beta) \tag{3.5}$$

then hold.

PROOF. Relations (3.2) and (3.3) can be written in the form

$$z - y_{zzz} + yy_{zz} - 3y_z^2 + 4y^2 y_z - y^4 = \frac{2 - 2\beta}{y(z, \beta) - y(z, 2-\beta)}, \tag{3.6}$$

$$z + 2y_{zzz} + 4yy_{zz} + 3y_z^2 - 2y^2 y_z - y^4 = \frac{2\beta + 1}{-y(z, \beta) + y(z, -1-\beta)}. \tag{3.7}$$

Replace β by $2-\beta$ in (3.6) and by $-1-\beta$ in (3.7). Equate the left parts of the expressions obtained and the left parts of (3.6) and (3.7). As a result we have equalities (3.4) and (3.5).

Substituting to remove y_{zzz} from (3.6) and (3.7), we get

$$\begin{aligned} 2yy_{zz} - y_z^2 + 2y^2 y_z - y^4 + z &+ \frac{4(\beta - 1)}{3(y(z, \beta) - y(z, 2-\beta))} \\ &+ \frac{2\beta + 1}{3(y(z, \beta) - y(z, -1-\beta))} = 0, \quad \beta \neq -1/2, 1. \end{aligned} \tag{3.8}$$

Let us introduce for the sake of convenience the following notation:

$$\begin{aligned} f &= y(z, \beta), & g &= y(z, 2-\beta), & h &= y(z, -1-\beta), \\ r &= y(z, \beta - 3), & p &= y(z, \beta + 3). \end{aligned}$$

One can write (3.4) and (3.5) in the form

$$g_z = f_z - f^2/2 + g^2/2, \tag{3.9}$$
$$h_z = f_z + f^2 - h^2. \tag{3.10}$$

From (3.9) and (3.10) we obtain

$$\begin{aligned} g_{zz} &= f_{zz} - f f_z + g f_z - f^2 g/2 + g^3/2, \\ g_z^2 &= f_z^2 - f^2 f_z + g^2 f_z + f^4/4 - f^2 g^2/2 + g^4/4, \\ h_{zz} &= f_{zz} + 2 f f_z - 2h f_z - 2 f^2 h + 2h^3, \\ h_z^2 &= f_z^2 + 2 f^2 f_z - 2h^2 f_z + f^4 - 2 f^2 h^2 + h^4. \end{aligned} \tag{3.11}$$

Relation (3.8) for f, g and h takes the form

$$2 f f_{zz} - f_z^2 + 2 f^2 f_z - f^4 + z + \frac{4}{3}\frac{\beta - 1}{f - g} + \frac{1}{3}\frac{2\beta + 1}{f - h} = 0, \tag{3.12}$$
$$2g g_{zz} - g_z^2 + 2g^2 g_z - g^4 + z + \frac{4}{3}\frac{\beta - 1}{f - g} - \frac{1}{3}\frac{2\beta - 5}{g - r} = 0, \tag{3.13}$$
$$2h h_{zz} - h_z^2 + 2h^2 h_z - h^4 + z - \frac{4}{3}\frac{\beta + 2}{h - p} + \frac{1}{3}\frac{2\beta + 1}{f - h} = 0. \tag{3.14}$$

Substitute to remove the second derivatives of g and h with respect to x in (3.13) and (3.14). Taking into account (3.9)–(3.11) we get

$$\begin{aligned} &2g f_{zz} - f_z^2 + f^2 f_z - 2g f f_z + 3g^2 f_z - \frac{1}{4} f^4 - \frac{3}{2} f^2 g^2 + \frac{3}{4} g^4 + z \\ &\quad + \frac{4}{3}\frac{\beta - 1}{f - g} - \frac{1}{3}\frac{2\beta - 5}{g - r} = 0, \end{aligned} \tag{3.15}$$
$$2h f_{zz} - f_z^2 - 2 f^2 f_z + 4h f f_z - f^4 + z - \frac{4}{3}\frac{\beta + 2}{h - p} + \frac{1}{3}\frac{2\beta + 1}{f - h} = 0. \tag{3.16}$$

Substitute f_{zz} from (3.12) into (3.15) and (3.16). Multiply the relation obtained by $f/(g - f)$ and $f/(h - f)$ accordingly to get

$$\begin{aligned} &f_z^2 - (f - 3g) f f_z + \frac{1}{4} f^4 - \frac{3}{4} f^3 g + \frac{3}{4} f^2 g^2 + \frac{3}{4} f g^3 - z - \frac{4}{3}\frac{\beta - 1}{f - g} \\ &\quad - \frac{1}{3}\frac{2\beta + 1}{f - h} + \frac{(2\beta + 1) f}{3(f - g)(f - h)} + \frac{(2\beta - 5) f}{3(f - g)(g - r)} = 0, \end{aligned} \tag{3.17}$$
$$\begin{aligned} &f_z^2 + 2 f^2 f_z + f^4 - z - \frac{4}{3}\frac{\beta - 1}{f - g} - \frac{2\beta + 1}{3(f - h)} \\ &\quad + \frac{4}{3}\frac{f(\beta - 1)}{(f - h)(f - g)} + \frac{4}{3}\frac{f(\beta + 2)}{(f - h)(h - p)} = 0. \end{aligned} \tag{3.18}$$

Substituting to remove f_z^2 from (3.17) and (3.18), we find f_z in the form

$$f_z = -\frac{1}{4}(f+g)^2 - \frac{2\beta-5}{9(f-g)^2(f-h)} - \frac{4}{9}\frac{\beta+2}{(f-h)(f-g)(h-p)} + \frac{2\beta-5}{9(f-g)^2(g-r)}. \tag{3.19}$$

Substitution of (3.19) into (3.18) leads to a recursion formula for the solutions of (3.1) in the form

$$\left[\frac{g^2}{4}+\frac{fg}{2}-\frac{3}{4}f^2+\frac{4}{9}\frac{\beta+2}{(f-h)(f-g)(h-p)}+\frac{2\beta-5}{9(f-g)^2}\left(\frac{1}{f-h}-\frac{1}{g-r}\right)\right]^2 -z-\frac{2\beta+1}{3(f-h)}+\frac{4}{3}\frac{h(\beta-1)}{(f-h)(f-g)}+\frac{4}{3}\frac{f(\beta+2)}{(f-h)(h-p)}=0. \tag{3.20}$$

Note that (3.20) is not valid for $\beta=-2,-1/2,1,5/2$. Equation (3.20) can be written substituting (3.19) into (3.17). However, calculating value r (or p) at the four given values of h, f, g and p (or r), one obtains two roots. One of them does not satisfy (3.1).

A unique solution can be found if the recursion formula for six solutions of (3.1) is used. Let us obtain this equation. Replace β by $-\beta-1$ in (3.19). In this case we have derivative h_z in the form

$$h_z = -\frac{1}{4}(h+p)^2 + \frac{2\beta+7}{9(h-p)^2(h-f)} + \frac{4}{9}\frac{\beta-1}{(h-f)(h-p)(f-g)} - \frac{2\beta+7}{9(h-p)^2(p-d)}, \tag{3.21}$$

where $d=y(z,-\beta-4)$. Taking into account the expressions (3.19), (3.21) and relation (3.10), we get the following recursion formula:

$$f^2-h^2+\frac{1}{4}(h+p)^2-\frac{1}{4}(f+g)^2-\frac{4}{3(f-h)(f-g)(h-p)} + \frac{2\beta-5}{9(f-g)^2}\left[\frac{1}{g-r}-\frac{1}{f-h}\right]+\frac{2\beta+7}{9(h-p)^2}\left[\frac{1}{p-d}+\frac{1}{f-h}\right]=0. \tag{3.22}$$

Equation (3.22) is not valid for $\beta=-7/2,-2,-1/2,1,5/2$. Note that (3.22) is invariant under the transformation $\beta\to-\beta-1$.

Replacing β by $-\beta+2$ in (3.19) we have the derivative g_z in the form

$$g_z=-\frac{1}{4}(f+g)^2-\frac{2\beta+1}{9(f-g)^2}\left[\frac{1}{f-h}-\frac{1}{g-r}\right]-\frac{4}{9}\frac{\beta-4}{(f-g)(g-r)(r-q)}, \tag{3.23}$$

where $q = y(z, -\beta + 5)$. Taking into account the expressions (3.19), (3.23) and relation (3.9), we get the following recursion formula:

$$f^2 - g^2 - \frac{4}{3(f-g)^2}\left[\frac{1}{f-h} - \frac{1}{g-r}\right] - \frac{8}{9}\frac{\beta - 4}{(f-g)(g-r)(r-q)} + \frac{8}{9}\frac{\beta + 2}{(f-g)(f-h)(h-p)} = 0. \tag{3.24}$$

Equation (3.24) is not valid for $\beta = -2, -1/2, 1, 5/2, 4$. The last equation is invariant under the transformation $\beta \to -\beta + 2$.

We cannot introduce a discrete equation for one variable from (3.22) (or (3.24)) because there are positive and negative values of parameter β. However, assuming $y_n = y(z, \beta_n)$, $x_n = y(z, -\beta_n - 1)$, $\beta_{n+1} = \beta_n + 3$, one can find the system of discrete equations from (3.22) and (3.24). It takes the form

$$\begin{aligned} y_n^2 - x_{n-1}^2 &- \frac{4}{3(y_n - x_{n-1})^2}\left(\frac{1}{y_{n-1} - x_{n-1}} + \frac{1}{y_n - x_n}\right) \\ &+ \frac{8}{9}\frac{\beta_n - 4}{(y_{n-1} - x_{n-2})(y_{n-1} - x_{n-1})(y_n - x_{n-1})} \\ &- \frac{8}{9}\frac{\beta_n + 2}{(y_n - x_{n-1})(y_n - x_n)(y_{n+1} - x_n)} = 0, \end{aligned} \tag{3.25}$$

$$\begin{aligned} y_n^2 - x_n^2 &+ \frac{1}{4}(y_{n+1} + x_n)^2 - \frac{1}{4}(y_n + x_{n-1})^2 \\ &+ \frac{4}{3(y_n - x_{n-1})(y_n - x_n)(y_{n+1} - x_n)} \\ &- \frac{2\beta_n - 5}{9(y_n - x_{n-1})^2}\left(\frac{1}{y_{n-1} - x_{n-1}} + \frac{1}{y_n - x_n}\right) \\ &+ \frac{2\beta_n + 7}{9(y_{n+1} - x_n)^2}\left(\frac{1}{y_n - x_n} + \frac{1}{y_{n+1} - x_{n+1}}\right) = 0. \end{aligned} \tag{3.26}$$

This system is not valid for $\beta_n = -7/2, -2, -1/2, 1, 5/2, 4$. The system of equations (3.25), (3.26) can be used for finding solutions y_{n+1} and x_{n+1}, given x_{n-2}, x_{n-1}, x_n, y_{n-1} and y_n.

Thus the results of this work show that the method used in [4, 7, 10, 25] can be applied to finding the discrete equation corresponding to the fourth-order differential equations of the P_2 and K_2 hierarchies. These equations are given by (2.20), (3.25) and (3.26).

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References

[1] H. Airault, "Rational solutions of Painlevé equations", *Stud. Appl. Math.* **61** (1979) 31–53.

[2] M. Boiti and F. Pempinelli, "Similarity solutions of the Korteweg-de Vries equation", *Nuovo Cimento Soc. Ital. Fis.* **B51** (1979) 70–78.

[3] P. J. Caudrey, R. K. Dodd and J. D. Gibbon, "A new hierarchy of Korteweg-de Vries equations", *Proc. Roy. Soc. London* **A351** (1976) 407–422.

[4] P. Clarkson, E. L. Mansfield and H. N. Webster, "On the relation between discrete and continuous Painlevé equations", *Theoret. and Math. Phys.* **122** (2000) 1–16.

[5] R. K. Dodd and J. D. Gibbon, "The prolongation structure of a higher order Korteweg-de Vries equation", *Proc. Roy. Soc. London* **A358** (1978) 287–296.

[6] H. Flaschka and A. C. Newell, "Monodromy- and spectrum-preserving deformations. I", *Comm. Math. Phys.* **76** (1980) 65–116.

[7] A. S. Fokas, B. Grammaticos and A. Ramani, "From continuous to discrete Painlevé equations", *J. Math. Anal. Appl.* **180** (1993) 342–360.

[8] P. R. Gordoa, N. Joshi and A. Pickering, "Mappings preserving locations of movable poles: a new extension of the truncation method to ordinary differential equations", *Nonlinearity* **12** (1999) 955–1008.

[9] B. Grammaticos, F. W. Nijhoff and A. Ramani, "Discrete Painlevé equations", in *The Painlevé Property. One century later* (ed. R. Conte), CRM Series in Math. Phys., (Springer, New York, 1999) 413–516.

[10] B. Grammaticos and A. Ramani, "From continuous Painlevé IV to the asymmetric discrete Painlevé I", *J. Phys.* **A31** (1998) 5787–5798.

[11] V. I. Gromak, "Nonlinear evolution equations and equations of p type", *Differential equations* **20** (1984) 2042–2048, (Russian).

[12] V. I. Gromak and N. A. Lukashevich, *Analytical Properties of Painlevé equations* (Izdat. Universitetskoe, Minsk, 1990).

[13] D. J. Gross and A. A. Migdal, "Nonperturbative two-dimensional quantum gravity", *Phys. Rev. Lett.* **64** (1990) 127–130.

[14] D. J. Gross and A. A. Migdal, "A nonperturbative treatment of two-dimensional quantum gravity", *Nuclear Phys.* **B340** (1990) 333–365.

[15] A. N. W. Hone, "Non-autonomous Hénon-Heils systems", *Physica* **D118** (1998) 1–26.

[16] N. A. Kudryashov, "The first and the second Painlevé equations of higher order and some relations between them", *Phys. Lett.* **A224** (1997) 353–360.

[17] N. A. Kudryashov, "On new transcendents defined by nonlinear ordinary differential equations", *J. Phys. A.: Math. Gen.* **31** (1998) L129–L137.

[18] N. A. Kudryashov, "Transcendents defined by nonlinear fourth-order ordinary differential equations", *J. Phys. A.: Math. Gen.* **32** (1999) 999–1013.

[19] N. A. Kudryashov, "Two hierarchies of ordinary differential equations and their properties", *Phys. Lett.* **A252** (1999) 173–179.

[20] N. A. Kudryashov, "Fourth-order nonlinear differential equations with solutions in the form of transcendents", *Theoret. and Math. Phys.* **122** (2000) 58–71.

[21] N. A. Kudryashov and M. B. Soukharev, "Uniformization and transcendence of solutions for the first and second Painlevé hierarchies", *Phys. Lett.* **A237** (1998) 206–216.

[22] B. Kupershmidt and G. Wilson, "Modifying Lax equations and the second Hamiltonian structure", *Invent. Math.* **62** (1981) 403–436.

[23] N. A. Lukashevich, "On the theory of the second Painlevé equation", *Differential equations* **7** (1971) 1124–1125.

[24] F. W. Nijhoff and V. G. Papageorgiou, "Similarity reductions of integrable lattices and discrete

analogues of the Painlevé II equation”, *Phys. Lett.* **A153** (1991) 337–344.

[25] F. W. Nijhoff, J. Satsuma, K. Kajiwara, B. Grammaticos and A. Ramani, “A study of the alternate discrete Painlevé II equation”, *Inverse Problems* **12** (1996) 697–716.

[26] G. R. W. Quispel, J. A. G. Roberts and C. J. Thompson, “Integrable mappings and soliton equations”, *Phys. Lett.* **A126** (1988) 419–421.

[27] A. P. Veselov, “Integrable mappings”, *Russian Math. Surveys* **46** (1991) 3–45.

[28] J. Weiss, “On classes of integrable systems and the Painlevé property”, *J. Math. Phys.* **25** (1984) 13–24.

[29] J. Weiss, “Bäcklund transformation and the Painlevé property”, *J. Math. Phys.* **27** (1986) 1293–1305.