*ANZIAM J.* 43(2002), 429–447

# CALCULATION OF CHAKALOV-POPOVICIU QUADRATURES OF RADAU AND LOBATTO TYPE

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(Received 25 January, 1999; revised 19 April, 2001)

**Abstract**

A numerical method for calculation of the generalized Chakalov-Popoviciu quadrature formulae of Radau and Lobatto type, using the results given for the generalized Chakalov-Popoviciu quadrature formula, is given. Numerical results are included. As an application we discuss the problem of approximating a function $f$ on the finite interval $I = [a, b]$ by a spline function of degree $m$ and variable defects $d_\nu$, with $n$ (variable) knots, matching as many of the initial moments of $f$ as possible. An analytic formula for the coefficients in the generalized Chakalov-Popoviciu quadrature formula is given.

## 1. Introduction

Let $d\lambda(t)$ be a nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support supp $(d\lambda)$, for which all moments

$$\mu_k = \int_{\mathbb{R}} t^k \, d\lambda(t), \qquad k = 0, 1, \ldots,$$

exist and are finite, and $\mu_0 > 0$. A quadrature formula of the form

$$\int_{\mathbb{R}} f(t)\, d\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{1.1}$$

where $A_{i,\nu} = A_{i,\nu}^{G} = A_{i,\nu}^{(n,s)}$, $\tau_\nu = \tau_\nu^{(n,s)}$, which is exact for all algebraic polynomials of degree at most $2(s+1)n - 1$, was considered firstly by P. Turán (see [20]), in the case when $d\lambda(t) = dt$ on $[-1, 1]$. The case with a weight function, $d\lambda(t) = \omega(t)\, dt$ on the interval $[a, b]$, has been considered by the Italian mathematicians Ossicini, Ghizzetti,

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Guerra and Rosati, and also by Chakalov, Stroud, Stancu, Ionescu, Pavel, *etc*. (see [15] for references).

The nodes $\tau_\nu$ in (1.1) must be zeros of a (monic) polynomial $\pi_n(t)$ which minimizes the integral

$$F \equiv F(a_0, a_1, \ldots, a_{n-1}) = \int_{\mathbb{R}} \pi_n(t)^{2s+2}\, d\lambda(t),$$

where

$$\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

In order to minimize $F$ we must have

$$\int_{\mathbb{R}} \pi_n(t)^{2s+1} t^k\, d\lambda(t) = 0, \quad k = 0, 1, \ldots, n-1. \tag{1.2}$$

Polynomials $\pi_n(t)$ which satisfy this new type of orthogonality "*power orthogonality*" are known as $s$-orthogonal (or $s$-self associated) polynomials with respect to the measure $d\lambda(t)$.

For $s = 0$ we have the standard case of orthogonal polynomials.

Let $n \in N$ and let $\sigma = \sigma_n = (s_1, s_2, \ldots, s_n)$ be a sequence of nonnegative integers. A generalization of the Gauss-Turán quadrature formula (1.1) to rules having nodes with arbitrary multiplicities was given, independently, by Chakalov [2, 3] and Popoviciu [17].

In this case, it is important to assume that the nodes $\tau_\nu$ $(= \tau_\nu^{(n,\sigma)})$ are ordered, say

$$\tau_1 < \tau_2 < \cdots < \tau_n, \quad \tau_\nu \in \operatorname{supp}(d\lambda), \tag{1.3}$$

with odd multiplicities

$$2s_1 + 1,\ 2s_2 + 1,\ \ldots,\ 2s_n + 1,$$

respectively. Then the corresponding quadrature formula

$$\int_{\mathbb{R}} f(t)\, d\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{1.4}$$

where $A_{i,\nu} = A_{i,\nu}^G = A_{i,\nu}^{(n,\sigma)}$, $\tau_\nu = \tau_\nu^{(n,\sigma)}$, has the maximum degree of exactness

$$d_{\max} = 2\sum_{\nu=1}^{n} s_\nu + 2n - 1 \tag{1.5}$$

if and only if

$$\int_{\mathbb{R}} \prod_{\nu=1}^{n} (t - \tau_\nu)^{2s_\nu+1} t^k\, d\lambda(t) = 0, \quad k = 0, \ldots, n-1. \tag{1.6}$$

The last *orthogonality conditions* correspond to (1.2). The existence of such quadrature rules has been proved by Chakalov [2], Popoviciu [17] and Morelli and Verna [16] and existence and uniqueness subject to (1.3) by Ghizzetti and Ossicini [10].

The conditions (1.6) define a sequence of polynomials $\{\pi_{n,\sigma}\}_{n\in N_0}$,

$$\pi_{n,\sigma}(t)=\prod_{\nu=1}^{n}\left(t-\tau_\nu^{(n,\sigma)}\right),\qquad \tau_1^{(n,\sigma)}<\tau_2^{(n,\sigma)}<\cdots<\tau_n^{(n,\sigma)},\ \tau_\nu^{(n,\sigma)}\in\operatorname{supp}(d\lambda),$$

such that

$$\int_{\mathbb{R}}\pi_{k,\sigma}(t)\prod_{\nu=1}^{n}\left(t-\tau_\nu^{(n,\sigma)}\right)^{2s_\nu+1}d\lambda(t)=0,\qquad k=0,\dots,n-1.$$

These polynomials are called $\sigma$-orthogonal polynomials and they correspond to the sequence $\sigma=(s_1,s_2,\dots)$. We shall often write simply $\tau_\nu$ or $\tau_\nu^{(n)}$ instead of $\tau_\nu^{(n,\sigma)}$. If we have $\sigma=(s,s,\dots)$, the above polynomials reduce to the $s$-orthogonal polynomials.

An iterative process for computing the coefficients of $s$-orthogonal polynomials in a special case, when the interval $[a,b]$ is symmetric with respect to the origin and the weight $\omega$ (in the case $d\lambda(t)=\omega(t)\,dt$ on $[a,b]$) is an even function, was proposed by Vincenti [21]. He applied his process to the Legendre case. When $n$ and $s$ increase, the process becomes numerically unstable.

In [12] (see also [8]) a numerical procedure for stably calculating the nodes $\tau_\nu$ in (1.1) was proposed. In [8] a numerical procedure for stably calculating the coefficients $A_{i,\nu}$ in (1.1) was also proposed. Some alternative methods were proposed in [19, 11] and [14] (see also [18]). In [15] the methods from [8, 14] for calculating the coefficients $A_{i,\nu}$ in (1.1) were generalized to be able to handle those in (1.4). A simple numerical method for stably calculating the nodes $\tau_\nu$ in (1.4) has been considered recently in [13]. For all calculations in this paper we shall use the methods from [15, 13].

## 2. Quadrature formulae of Radau and Lobatto type connected to $\sigma$-orthogonal polynomials

Let $[a,b]$ be the support of the nonnegative measure $d\psi(t)=w(t)\,dt$, where $w(t)$ is the weight function.

Let

$$\int_a^b u(t)\,d\psi(t)=\sum_{k=0}^{p}\alpha_k u^{(k)}(a)+\sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu}A_{i,\nu}^R u^{(i)}(\tau_\nu)+R_{n,p}^R,\tag{2.1}$$

$\tau_\nu\in(a,b)$, $-\infty<a<\infty$, $p\in N_0$, with

$$R_{n,p}^R(u;d\psi)=0\quad\text{for } u\in\mathscr{P}_{2(\sum_{\nu=1}^{n}s_\nu+n)+p},$$

be the generalized Chakalov-Popoviciu quadrature formula of Radau type.

Let

$$\int_a^b u(t)\,d\psi(t)=\sum_{k=0}^{p}\alpha_k u^{(k)}(a)+\sum_{k=0}^{q}\beta_k u^{(k)}(b)+\sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu}A_{i,\nu}^{L}u^{(i)}(\tau_\nu)+R_{n,p,q}^{L}, \quad (2.2)$$

$\tau_\nu\in(a,b)$, $-\infty<a<b<\infty$, $p,q\in N_0$, with

$$R_{n,p,q}^{L}(u;d\psi)=0 \quad \text{for } u\in\mathscr{P}_{2(\sum_{\nu=1}^{n}s_\nu+n)+p+q+1},$$

be the generalized Chakalov-Popoviciu quadrature formula of Lobatto type.

With $\mathscr{P}_k$ we denote the set of all polynomials of degree at most $k$, $k\in N_0$.

By using the results of Ghizzetti and Ossicini [9], we shall prove the existence and the uniqueness of the formula (2.2).

We shall denote by $\mathscr{L}[a,b]$ the class of Lebesgue-integrable (summable) functions in $[a,b]$ and by $AC^k[a,b]$ the class of functions whose $k$-th derivative is absolutely continuous in $[a,b]$, $k=0,1,2,\dots$.

Let us consider in $[a,b]$ a *linear differential operator of order* $L$, $L=1,2,3,\dots$,

$$E=E_L=\sum_{k=0}^{L}a_k(t)\frac{d^{L-k}}{dt^{L-k}}$$

with the following conditions on the coefficients $a_k(t)$:

$$a_0(t)=1;\quad a_k(t)\in AC^{L-k-1}[a,b],\ k=1,2,\dots,L-1;\quad a_L(t)\in\mathscr{L}[a,b].$$

The operator $E$ can be applied to the functions $u(t)\in AC^{L-1}[a,b]$, obtaining the function (defined almost everywhere):

$$E[u(t)]=E(u)=\sum_{k=0}^{L}a_k(t)u^{(L-k)}(t)\in\mathscr{L}[a,b].$$

We associate with the operator $E$ the *reduced operators*

$$E_r=\sum_{k=0}^{r}a_k(t)\frac{d^{r-k}}{dt^{r-k}},\quad r=0,1,\dots,L-1,$$

and their so-called *adjoint operators*

$$E_r^*=\sum_{k=0}^{r}(-1)^{r-k}\frac{d^{r-k}}{dt^{r-k}}a_k(t),\quad r=0,1,\dots,L,$$

where $E_L^*=E^*$.

Let $K(t,\xi)$ be the so-called *Cauchy resolvent kernel*, which is (as a function of $t$) the particular solution of the homogeneous equation $E(u)=0$ which satisfies, at the point $\xi$, the initial conditions:

$$\left[\frac{\partial^h}{\partial t^h K(t,\xi)}\right]_{t=\xi}=\delta_{h,L-1},\quad h=0,1,\dots,L-1,$$

where

$$\delta_{rs} = \begin{cases} 0, & r \neq s \\ 1, & r = s. \end{cases}$$

Let us consider the elementary quadrature formula

$$\int_a^b u(t)\, d\psi(t) = \sum_{h=0}^{L-1} \sum_{i=1}^{l} C_{hi} u^{(h)}(x_i) + R(u), \quad [E(u) = 0 \Rightarrow R(u) = 0], \tag{2.3}$$

where $E$ is the linear differential operator of order $L$.

In [9, pp. 29–31] the following result is proved.

THEOREM 2.1. *If, having $l$ fixed nodes $x_1, x_2, \ldots, x_l$ and $lL$ constants $C_{hi}$, the linear functional*

$$R(u) = \int_a^b u(t) w(t)\, dt - \sum_{h=0}^{L-1} \sum_{i=1}^{l} C_{hi} u^{(h)}(x_i)$$

*is null when $u$ is a solution of the homogeneous linear differential equation $E(u) = 0$, then there are $l-1$ uniquely determined solutions $\varphi_1(t), \ldots, \varphi_{l-1}(t)$ of the differential equation $E^*(\varphi) = w$ which, together with $\varphi_0(t)$ and $\varphi_l(t)$ given by*

$$\varphi_0(t) = -\int_a^t K(\xi, t) w(\xi)\, d\xi, \quad \varphi_l(t) = \int_t^b K(\xi, t) w(\xi)\, d\xi,$$

*validate*

$$C_{hi} = \{E^*_{L-h-1}[\varphi_i(t) - \varphi_{i-1}(t)]\}_{t=x_i}; \quad h = 0, 1, \ldots, L-1, \; i = 1, 2, \ldots, l,$$

*and*

$$R[u(t)] = \sum_{i=0}^{l} \int_{x_i}^{x_{i+1}} \varphi_i(t) E[u(t)]\, dt.$$

Having fixed the nodes $x_1, x_2, \ldots, x_l$ and the linear differential operator $E$, we may write the quadrature formula (2.3) in $\infty^{(l-1)L}$ different ways, since $(l-1)L$ is the number of arbitrary constants on which the $l-1$ solutions $\varphi_1(t), \ldots, \varphi_{l-1}(t)$ of the differential equation $E^*(\varphi) = w$ of order $L$ depend.

**Define the generalized Gauss problem** (see [9, pp. 41–45]).

The question is whether, having fixed nonnegative integers $p_i$ ($p_i \leq L-1$), $i = 1, \ldots, l$, with $(\exists\, i = 1, \ldots, l)$ $p_i \geq 1$, it is possible to make use of the arbitrary nature of these parameters to drop the values $u^{(h)}(x_i)$ of the derivatives of order higher than $L - p_i - 1$, $i = 1, \dots, l$, from (2.3), that is, whether there can exist a formula of the type

$$\int_a^b u(t)\,d\psi(t) = \sum_{i=1}^{l} \sum_{h=0}^{L-p_i-1} C_{hi} u^{(h)}(x_i) + R(u), \quad [E(u) = 0 \Rightarrow R(u) = 0]. \tag{2.4}$$

The answer is given by the following theorem (see [9, Problem 2, p. 45]), which can be proved similarly to Theorem 2.5.I in [9].

THEOREM 2.2. *Given the nodes $x_1, \dots, x_l$, which satisfy*

$$a \le x_1 < x_2 < \cdots < x_l \le b, \tag{2.5}$$

*the linear differential operator $E$ of order $L$ and nonnegative integers $p_i$ $(p_i \le L-1)$, $i = 1, \dots, l$, with $(\exists\, i = 1, \dots, l)$ $p_i \ge 1$, consider the homogeneous differential problem*

$$E(u) = 0; \quad u^{(h)}(x_i) = 0, \quad h = 0, 1, \dots, L - p_i - 1, \; i = 1, \dots, l. \tag{2.6}$$

*If this problem has no non-trivial solutions [whence $L \le lL - \sum_{i=1}^{l} p_i$] it is possible to write a quadrature formula of the type (2.4) in $\infty^{lL - \sum_{i=1}^{l} p_i - L}$ different ways. If on the other hand the problem (2.6) has $q$ linearly independent solutions $U_j(t)$ [$j = 1, 2, \dots, q$, with $L - Ll + \sum_{i=1}^{l} p_i \le q \le p_i$ $(\forall i = 1, \dots, l)$; $1 \le q$] then (2.4) may apply only if the $q$ conditions*

$$\int_a^b U_j(t)\,d\psi(t) = 0, \quad j = 1, \dots, q,$$

*are satisfied; if so, there are $\infty^{lL - \sum_{i=1}^{l} p_i - L + q}$ possible formulae of form (2.4).*

Consider (2.2), with conditions (2.5) for

$$x_1 = a, \quad x_{\nu+1} = \tau_\nu, \; \nu = 1, \dots, n, \quad x_l = x_{n+2} = b,$$
$$(\text{where} \quad C_{h1} = \alpha_h, \; C_{hi} = A_{h,i}^{L}, \; C_{hl} = C_{h,n+2} = \beta_h)$$

for which $R(u) = 0$, $\forall u \in \mathscr{P}_{2(\sum_{\nu=1}^{n} s_\nu + n) + p + q + 1}$.

Let $L = 2\big(\sum_{\nu=1}^{n} s_\nu + n\big) + p + q + 2$. By virtue of Theorem 2.2 we must consider the boundary problem

$$d^L u / dt^L = 0;$$

with

$$u^{(h)}(a) = 0, \quad h = 0, \dots, p; \quad u^{(h)}(b) = 0, \quad h = 0, \dots, q;$$
$$u^{(h)}(\tau_\nu) = 0, \quad h = 0, \dots, 2s_\nu, \; \nu = 1, \dots, n,$$

and its non trivial solutions which are

$$t^k(t-a)^{p+1}(b-t)^{q+1}\prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1},\quad k=0,1,\dots,n-1.$$

Therefore, (2.2) is possible if and only if

$$\int_a^b (t-a)^{p+1}(b-t)^{q+1}\cdot t^k\prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}\,d\psi(t)=0,\quad k=0,1,\dots,n-1,$$

are satisfied and this shows that the nodes $\tau_\nu$ must coincide with the zeros of the polynomial $\pi_{n,\sigma}(t)$ of the $\sigma$-orthogonal system relative to the measure

$$(t-a)^{p+1}(b-t)^{q+1}\,d\psi(t).$$

With such a choice of the nodes (2.2) is unique since, with the notation of Theorem 2.2, we have

$$lL-\sum_{i=1}^{l}p_i-L+q=p+q+2+\sum_{\nu=1}^{n}(2s_\nu+1)$$
$$-\left[2\left(\sum_{\nu=1}^{n}s_\nu+n\right)+p+q+2\right]+n=0.$$

Similarly, we can conclude that (2.1) exists and it is necessarily unique. In the following, we shall put $p=m=q$, without loss of generality.

## 3. Calculation of the formulae (2.1), (2.2)

We give two lemmas, which give a connection between the generalized Chakalov-Popoviciu quadrature (1.4) and the corresponding formulae of Radau and Lobatto type.

LEMMA 3.1. *If the measure $d\psi(t)$ admits** *the generalized Chakalov-Popoviciu quadrature of Lobatto type* (2.2) (*in which $p=q=m$*), *with distinct real zeros $\tau_\nu=\tau_\nu^{(n)}=\tau_\nu^{(n,\sigma)}$, $\nu=1,\dots,n$, all contained in the open interval $(a,b)$, there exists then a generalized Chakalov-Popoviciu formula*

$$\int_a^b g(t)\,d\lambda(t)=\sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu}A_{i,\nu}^{G}g^{(i)}\left(\tau_\nu^{(n)}\right)+R_n^G(g),\tag{3.1}$$

*For example, this holds if $d\psi(t)$ is nonnegative (or nonpositive).

*where* $d\lambda(t) = [(b-t)(t-a)]^{m+1}\, d\psi(t)$*, the nodes* $\tau_\nu^{(n)}$ *are the zeros of* $\sigma$*-orthogonal polynomial* $\pi_{n,\sigma}(\cdot;\, d\lambda)$*, while the weights* $A_{i,\nu}^G$ *are expressible in terms of those in* (2.2) *by*

$$A_{i,\nu}^G = \sum_{k=i}^{2s_\nu} \binom{k}{i} \left[((b-t)(t-a))^{m+1}\right]_{t=\tau_\nu}^{(k-i)} A_{k,\nu}^L, \tag{3.2}$$

*where* $i = 0, \ldots, 2s_\nu$, $\nu = 1, \ldots, n$.

PROOF. Let $g(t) = ((b-t)(t-a))^{m+1} p(t)$, $p \in \mathscr{P}_{2(\sum_{\nu=1}^n s_\nu + n)-1}$ and $\tau_\nu = \tau_\nu^{(n)}$. We have by (2.2)

$$\int_a^b g(t)\, d\psi(t) = \sum_{\nu=1}^{n} \sum_{k=0}^{2s_\nu} \left[((b-t)(t-a))^{m+1} p(t)\right]_{t=\tau_\nu}^{(k)} A_{k,\nu}^L,$$

and by (3.1)

$$\int_a^b p(t)\, d\lambda(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu}^G p^{(i)}(\tau_\nu).$$

So, we have that

$$\sum_{\nu=1}^{n} \sum_{k=0}^{2s_\nu} \left[((b-t)(t-a))^{m+1} p(t)\right]_{t=\tau_\nu}^{(k)} A_{k,\nu}^L = \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu}^G p^{(i)}(\tau_\nu).$$

Applying the Leibniz formula to the $k$-th derivative in the second sum, we find

$$\begin{aligned}
&\sum_{k=0}^{2s_\nu} \left[((b-t)(t-a))^{m+1} p(t)\right]_{t=\tau_\nu}^{(k)} A_{k,\nu}^L \\
&\quad = \sum_{k=0}^{2s_\nu} \left[ \sum_{i=0}^{k} \binom{k}{i} \left(((b-t)(t-a))^{m+1}\right)^{(k-i)} p^{(i)}(t) \right]_{t=\tau_\nu} A_{k,\nu}^L \\
&\quad = \sum_{i=0}^{2s_\nu} \left( \sum_{k=i}^{2s_\nu} \binom{k}{i} \left(((b-t)(t-a))^{m+1}\right)_{t=\tau_\nu}^{(k-i)} A_{k,\nu}^L p^{(i)}(\tau_\nu) \right) = \sum_{i=0}^{2s_\nu} A_{i,\nu}^G p^{(i)}(\tau_\nu),
\end{aligned}$$

where

$$A_{i,\nu}^G = \sum_{k=i}^{2s_\nu} \binom{k}{i} \left[((b-t)(t-a))^{m+1}\right]_{t=\tau_\nu}^{(k-i)} A_{k,\nu}^L; \quad i = 0, \ldots, 2s_\nu,\ \nu = 1, \ldots, n.$$

Similarly we can prove the following lemma.

LEMMA 3.2. *If the measure* $d\psi(t)$ *admits the generalized Chakalov-Popoviciu quadrature of Radau type* (2.1) *(in which* $p = m$*), with distinct real zeros* $\tau_\nu = \tau_\nu^{(n)*}$,

TABLE 4.1.

| $\nu$ | $\tau_{2\nu-1}$ | $\tau_{2\nu}$ |
|---|---|---|
| 1 | 8.06063896919729(−02) | 2.42198578093389(−01) |
| 2 | 4.93117605175704(−01) | 7.15377067743040(−01) |
| 3 | 8.94837669670698(−01) | |

*$\nu = 1, \dots, n$, all contained in the open interval $(a, b)$, there exists then a generalized Chakalov-Popoviciu formula (3.1), where $d\lambda(t) = d\lambda^*(t) = (t-a)^{m+1}\, d\psi(t)$, the nodes $\tau_\nu^{(n)*}$ are the zeros of $\sigma$-orthogonal polynomial $\pi_{n,\sigma}(\cdot; d\lambda^*)$, while the weights $A_{i,\nu}^G$ are expressible in terms of those in (3.1) by*

$$A_{i,\nu}^G = \sum_{k=i}^{2s_\nu} \binom{k}{i} \left[(t-a)^{m+1}\right]_{t=\tau_\nu}^{(k-i)} A_{k,\nu}^R; \quad i = 0, \dots, 2s_\nu, \ \nu = 1, \dots, n. \tag{3.3}$$

We can write the triangular system (3.2) in the form

$$A_{i,\nu}^G = \sum_{k=i}^{2s_\nu} C_k^{(i,\nu)} A_{k,\nu}^L; \quad i = 0, \dots, 2s_\nu, \ \nu = 1, \dots, n,$$

where

$$\begin{aligned} C_k^{(i,\nu)} &= \binom{k}{i} \left[((b-t)(t-a))^{m+1}\right]_{t=\tau_\nu}^{(k-i)} \\ &= \begin{cases} 0; & k < i, \\ \dfrac{k!}{i!} \displaystyle\sum_{l=0}^{k-i} \dfrac{(-1)^l (m+1)!^2 (\tau_\nu - a)^{m-k+i+l+1} (b-\tau_\nu)^{m-l+1}}{l!(k-i-l)!(m-k+i+l+1)!(m-l+1)!}; & i \le k \le 2s_\nu. \end{cases} \end{aligned}$$

The triangular system (3.3) we can write in the form

$$A_{i,\nu}^G = \sum_{k=i}^{2s_\nu} B_k^{(i,\nu)} A_{k,\nu}^R; \quad i = 0, \dots, 2s_\nu, \ \nu = 1, \dots, n,$$

where

$$B_k^{(i,\nu)} = \binom{k}{i} \left[(t-a)^{m+1}\right]_{t=\tau_\nu}^{(k-i)} = \begin{cases} 0; & k < i, \\ \dfrac{k!(m+1)!(\tau_\nu - a)^{m-k+i+1}}{i!(k-i)!(m-k+i+1)!}; & i \le k \le 2s_\nu. \end{cases}$$

## 4. Numerical results

As an example we consider the Chebyshev measure $d\psi(t) = dt/\sqrt{t-t^2}$ on the interval $I = [a, b] = [0, 1]$ in the Lobatto case. Therefore we have

$$d\lambda(t) = [t(1-t)]^{m+1/2}\, dt.$$

In Table 4.1 the nodes $\tau_\nu$ of the corresponding Chakalov-Popoviciu quadrature formula (1.4), for $\sigma = (0, 3, 1, 2, 1)$, $n = 5$, are given.

TABLE 4.2.

| $\nu$ | $i$ | $A^G_{i,\nu}$ | $A^G_{i+1,\nu}$ |
|---|---|---|---|
| 1 | 0 | 4.20127478080609(−08) | |
| 2 | 0 | 3.71485589869411(−05) | 2.53189264911106(−06) |
| 2 | 2 | 1.24288590234291(−07) | 3.28295940614803(−09) |
| 2 | 4 | 6.72398482227105(−11) | 7.51024105924184(−13) |
| 2 | 6 | 6.18123581366015(−15) | |
| 3 | 0 | 9.25967832748324(−05) | 1.88049797773032(−08) |
| 3 | 2 | 9.57294036599511(−08) | |
| 4 | 0 | 4.27128390332233(−05) | −1.71275165622089(−06) |
| 4 | 2 | 7.93022775662744(−08) | −1.08954169181538(−09) |
| 4 | 4 | 1.92447787210554(−11) | |
| 5 | 0 | 5.22053028280481(−07) | −1.15793712000017(−08) |
| 5 | 2 | 1.12436028390154(−10) | |

In Table 4.2 the weights $A^G_{i,\nu}$ of the corresponding Chakalov-Popoviciu quadrature formula are given. For $m = 5$, the weights $A^L_{i,\nu}$ of the corresponding Chakalov-Popoviciu quadrature formula of Lobatto type (2.2) are given in Table 4.3.

TABLE 4.3.

| $\nu$ | $i$ | $A^L_{i,\nu}$ | $A^L_{i+1,\nu}$ |
|---|---|---|---|
| 1 | 0 | 2.53603580873942(−01) | |
| 2 | 0 | 6.54607056346764(−01) | 2.47009978449190(−03) |
| 2 | 2 | 1.78916012822395(−03) | 8.68913193385365(−06) |
| 2 | 4 | 1.06575641867557(−06) | 3.29355080757672(−09) |
| 2 | 6 | 1.61701214701959(−10) | |
| 3 | 0 | 3.98578546685041(−01) | −1.82300441012789(−04) |
| 3 | 2 | 3.92553687612449(−04) | |
| 4 | 0 | 5.24003817562713(−01) | −8.43880698485214(−04) |
| 4 | 2 | 9.30751562588805(−04) | −1.57766077104084(−06) |
| 4 | 4 | 2.70074453090533(−07) | |
| 5 | 0 | 4.11726824044766(−01) | −3.70334318380999(−04) |
| 5 | 2 | 1.61911889209916(−04) | |

Table 4.4 gives the corresponding coefficients $\alpha_k$, $\beta_k$ in the endpoints $-1$, $1$. The numbers in parentheses denote decimal exponents. The programs were realized in double precision arithmetic in FORTRAN.

TABLE 4.4.

| $k$ | $\alpha_k$ | $\beta_k$ |
|---|---|---|
| 0 | 4.48079461557622(−01) | 4.50993366518945(−01) |
| 1 | 6.76966763724565(−03) | −6.86234369124486(−03) |
| 2 | 7.83092608702163(−05) | 7.94301775592061(−05) |
| 3 | 5.74636703570962(−07) | −5.80256392257038(−07) |
| 4 | 2.44687263671571(−09) | 2.45051821492370(−09) |
| 5 | 4.67095320822040(−12) | −4.62776252162197(−12) |

TABLE 4.5.

| $n$ | $\sigma$ | $m$ | $Re$ |
|---|---|---|---|
| 2 | (1, 1) | 0 | 1.0(−09) |
| 2 | (0, 2) | 1 | 3.6(−12) |
| 2 | (0, 3) | 1 | 9.9(−15) |
| 3 | (1, 0, 1) | 0 | 1.6(−12) |
| 3 | (0, 1, 2) | 0 | 4.8(−15) |
| 3 | (0, 1, 2) | 1 | 6.6(−16) |

By using (2.2) and the presented methods we have calculated the integral

$$J = \int_0^1 \frac{e^{2t}}{\sqrt{t - t^2}}\, dt = 10.8118661043980\ldots,$$

for some $n, \sigma, m$. In Table 4.5 the relative errors $Re$ of these calculations are given.

## 5. An application—Moment-preserving spline approximation with variable defects on finite intervals

Let $z_+^i$ be $z^i$, if $z \geq 0$, and 0, if $z < 0$.

In this section we discuss the case of approximating a function $f = f(t)$ on some given finite interval $I = [a, b]$, which can be standardized to $[a, b] = [0, 1]$, by a spline function of degree $m \geq 2$ and defects $d_\nu$ $(1 \leq d_\nu \leq m,\ \nu = 1, \ldots, n)$, with $n$ knots. Under suitable assumptions on $f$ and $d_\nu = 2s_\nu + 1$, $\nu = 1, \ldots, n$, we shall show that our problem has a unique solution if and only if certain generalized Chakalov-Popoviciu quadrature formulae of Radau and Lobatto type exist corresponding to measures depending on $f$. Existence, uniqueness and pointwise convergence are assured if $f$ is completely monotonic on $[0, 1]$.

**Spline approximation on [0, 1].** A spline function of degree $m \geq 2$ and defects $d_\nu$, $\nu = 1, \ldots, n$, with $n$ (distinct) knots $\tau_1, \ldots, \tau_n$ in the interior of $[0, 1]$, can be written in terms of truncated powers in the form

$$s_{n,m}(t) = p_m(t) + \sum_{\nu=1}^{n} \sum_{i=m-d_\nu+1}^{m} a_{i,\nu}(\tau_\nu - t)_+^i, \tag{5.1}$$

where $a_{i,\nu}$ are real numbers and $p_m(t)$ is a polynomial of degree $\le m$.

Similarly as in [5] we shall consider two related problems.

PROBLEM I. Determine $s_{n,m}$ in (5.1) such that

$$\int_0^1 t^j s_{n,m}(t)\,dt = \int_0^1 t^j f(t)\,dt, \quad j = 0, 1, \ldots, \sum_{\nu=1}^{n} d_\nu + n + m. \tag{5.2}$$

PROBLEM I*. Determine $s_{n,m}$ in (5.1) such that

$$s_{n,m}^{(k)}(1) = p_m^{(k)}(1) = f^{(k)}(1), \quad k = 0, \ldots, m, \tag{5.3}$$

and such that (5.2) holds for $j = 0, 1, \ldots, \sum_{\nu=1}^{n} d_\nu + n - 1$.

In this section we shall reduce our problems to $\sigma$-orthogonality and generalized Chakalov-Popoviciu quadratures by restricting the class of functions $f$.

In order to reduce our problems (5.2) and (5.3) to $\sigma$-orthogonality, we have to put $d_\nu = 2s_\nu + 1$, $\nu = 1, \ldots, n$, that is, the defects of the spline function (5.1) should be odd.

Let

$$\varphi_k = \frac{(-1)^k}{m!} f^{(k)}(1), \quad b_k = \frac{(-1)^k}{m!} p_m^{(k)}(1), \quad k = 0, \ldots, m. \tag{5.4}$$

Applying $m + 1$ integration by parts to the integrals in the moment equation (5.2) we obtain (see [5])

$$\begin{aligned}\sum_{k=0}^{m} b_k \left[t^{m+1+j}\right]_{t=1}^{(m-k)} + \sum_{\nu=1}^{n} \sum_{i=m-2s_\nu}^{m} a_{i,\nu}\tau_\nu^{j+i+1} \frac{i!(m+j+1)!}{m!(j+i+1)!} \\ = \sum_{k=0}^{m} \varphi_k \left[t^{m+1+j}\right]_{t=1}^{(m-k)} + \frac{(-1)^{m+1}}{m!} \int_0^1 t^{m+1+j} f^{(m+1)}(t)\,dt,\end{aligned} \tag{5.5}$$

where $j = 0, 1, \ldots, 2\big(\sum_{\nu=1}^{n} s_\nu + n\big) + m$.

For the second sum in (5.5) we may observe that

$$\sum_{\nu=1}^{n} \sum_{i=m-2s_\nu}^{m} a_{i,\nu}\tau_\nu^{j+i+1} \frac{i!(m+j+1)!}{m!(j+i+1)!} = \sum_{\nu=1}^{n} \sum_{i=m-2s_\nu}^{m} \frac{i!}{m!} a_{i,\nu} \left[t^{m+j+1}\right]_{t=\tau_\nu}^{(m-i)}.$$

Changing indices ($k = m - i$), the second sum on the right becomes

$$\sum_{k=0}^{2s_\nu} \frac{(m-k)!}{m!} a_{m-k,\nu} \left[t^{m+1} t^j\right]_{t=\tau_\nu}^{(k)}, \tag{5.6}$$

hence defining the measure

$$d\psi(t) = \frac{(-1)^{m+1}}{m!} f^{(m+1)}(t)\, dt \quad \text{on} \quad [0,1]. \tag{5.7}$$

Equation (5.5) may be rewritten

$$\sum_{k=0}^{m} b_k \left[t^{m+1+j}\right]_{t=1}^{(m-k)} + \sum_{\nu=1}^{n} \sum_{k=0}^{2s_\nu} \frac{(m-k)!}{m!} a_{m-k,\nu} \left[t^{m+1+j}\right]_{t=\tau_\nu}^{(k)}$$
$$= \sum_{k=0}^{m} \varphi_k \left[t^{m+1+j}\right]_{t=1}^{(m-k)} + \int_0^1 t^{m+1+j}\, d\psi(t), \tag{5.8}$$

where $j = 0, 1, \dots, 2\left(\sum_{\nu=1}^{n} s_\nu + n\right) + m$.

Now we can state the main result for Problem I.

THEOREM 5.1. *Let* $f \in C^{m+1}[0,1]$. *There exists a unique spline function* (5.1) *on* $[0,1]$, *with* $d_\nu = 2s_\nu + 1$, $\nu = 1, \dots, n$, *satisfying* (5.2) *if and only if the measure* $d\psi(t)$ *in* (5.7) *admits a generalized Chakalov-Popoviciu quadrature of Lobatto type*

$$\int_0^1 g(t)\, d\psi(t) = \sum_{k=0}^{m} [\alpha_k g^{(k)}(0) + \beta_k g^{(k)}(1)]$$
$$+ \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu}^{L} g^{(i)}(\tau_\nu^{(n)}) + R_{n,m}^{L}(g; d\psi), \tag{5.9}$$

*where*

$$R_{n,m}^{L}(g; d\psi) = 0 \quad \textit{for} \quad g \in \mathscr{P}_{2(\sum_{\nu=1}^{n} s_\nu + n + m)+1}, \tag{5.10}$$

*with distinct real zeros* $\tau_\nu^{(n)}$, $\nu = 1, \dots, n$, *all contained in the open interval* $(0,1)$. *The spline function in* (5.1) *is given by*

$$\tau_\nu = \tau_\nu^{(n)}, \quad a_{m-k,\nu} = \frac{m!}{(m-k)!} A_{k,\nu}^{L}; \quad \nu = 1, \dots, n, \ k = 0, \dots, 2s_\nu, \tag{5.11}$$

*where* $\tau_\nu^{(n)}$ *are the interior nodes of the generalized Chakalov-Popoviciu quadrature formula of Lobatto type and* $A_{k,\nu}^{L}$ *are the corresponding weights, while the polynomial* $p_m(t)$ *is given by*

$$p_m^{(k)}(1) = f^{(k)}(1) + (-1)^k m! \beta_{m-k}, \quad k = 0, 1, \dots, m, \tag{5.12}$$

*where* $\beta_{m-k}$ *is the coefficient of* $g^{(m-k)}(1)$ *in* (5.9).

PROOF. Putting $g(t) = t^{m+1}p(t)$, $p \in \mathscr{P}_{2(\sum_{\nu=1}^{n} s_\nu+n)+m}$, in (5.9) and noting (5.10) yields, for every $p \in \mathscr{P}_{2(\sum_{\nu=1}^{n} s_\nu+n)+m}$,

$$\sum_{k=0}^{m} \beta_k \left[t^{m+1}p(t)\right]_{t=1}^{(k)} + \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} A_{i,\nu}^{L} \left[t^{m+1}p(t)\right]_{t=\tau_\nu}^{(k)} = \int_0^1 t^{m+1}p(t)\,d\psi(t),$$

which is identical to (5.8), if we identify

$$b_{m-k} - \varphi_{m-k} = \beta_k, \qquad k = 0, 1, \dots, m;$$
$$a_{m-k,\nu} = \frac{m!}{(m-k)!} A_{k,\nu}^{L}, \quad \nu = 1, \dots, n,\ k = 0, \dots, 2s_\nu.$$

REMARK A. The case $s_1 = \cdots = s_n = 0$ of Theorem 5.1 has been obtained in [5], and generalized in [6] to the case $s_1 = \cdots = s_n = s$, $s \in N$.

If $f$ is completely monotonic on $[0, 1]$ then $d\psi(t)$ in (5.7) is a positive measure for every $m$, and then by virtue of the assumptions in Theorem 5.1 the generalized Chakalov-Popoviciu quadrature formula of Lobatto type exists uniquely, with $n$ distinct real nodes $\tau_\nu^{(n)}$ in $(0, 1)$.

The solution of Problem I* can be given in a similar way.

THEOREM 5.2. *Let $f \in C^{m+1}[0, 1]$. There exists a unique spline function on $[0, 1]$,*

$$s_{n,m}^{*}(t) = p_m^{*}(t) + \sum_{\nu=1}^{n}\sum_{i=m-2s_\nu}^{m} a_{i,\nu}^{*}(\tau_\nu^{*} - t)_{+}^{i}, \qquad \begin{array}{l} 0 < \tau_\nu^{*} < 1, \\ \tau_\nu^{*} \neq \tau_\mu^{*} \text{ for } \nu \neq \mu, \end{array} \tag{5.13}$$

*satisfying (5.3) and (5.2), for $j = 0, 1, \dots, 2\left(\sum_{\nu=1}^{n} s_\nu + n\right) - 1$, if and only if the measure $d\psi(t)$ in (5.7) admits a generalized Chakalov-Popoviciu quadrature of Radau type*

$$\int_0^1 g(t)\,d\psi(t) = \sum_{k=0}^{m} \alpha_k^{*} g^{(k)}(0) + \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} A_{i,\nu}^{R} g^{(i)}(\tau_\nu^{(n)*}) + R_{n,m}^{R}(g; d\psi), \tag{5.14}$$

*where*

$$R_{n,m}^{R}(g; d\psi) = 0 \quad \text{for } g \in \mathscr{P}_{2(\sum_{\nu=1}^{n} s_\nu+n)+m},$$

*with distinct real zeros $\tau_\nu^{(n)*}$, $\nu = 1, \dots, n$, all contained in the open interval $(0, 1)$. The knots $\tau_\nu^{*}$ in (5.13) are then precisely these zeros,*

$$\tau_\nu^{*} = \tau_\nu^{(n)*}, \quad \nu = 1, \dots, n, \tag{5.15}$$

*and*

$$a^*_{m-k,\nu} = \frac{m!}{(m-k)!} A^R_{k,\nu}; \quad \nu = 1,\dots,n, \; k = 0,\dots,2s_\nu, \tag{5.16}$$

*while the polynomial* $p^*_m(t)$ *is given by*

$$p^*_m(t) = \sum_{k=0}^{m} \frac{f^{(k)}(1)}{k!}(t-1)^k. \tag{5.17}$$

REMARK B. Therefore, by using our methods from [15, 13], the results from Section 3, and the formulae (5.11) and (5.12), or (5.15)–(5.17), we can easily determine the spline approximation $s_{n,m}(t)$, or $s^*_{n,m}(t)$, respectively.

**Error analysis.** Similarly as in [5], following [7], we can prove the following statement regarding the error of spline approximations.

THEOREM 5.3. *Define* $r_x(t) = (t-x)^m_+$, $0 \le t \le 1$. *Under the conditions of Theorems 5.1 and 5.2, we have*

$$f(x) - s_{n,m}(x) = R^L_{n,m}(r_x; d\psi), \quad 0 < x < 1, \tag{5.18}$$

*and*

$$f(x) - s^*_{n,m}(x) = R^R_{n,m}(r_x; d\psi), \quad 0 < x < 1, \tag{5.19}$$

*respectively, where* $R^L_{n,m}(g; d\psi)$ *and* $R^R_{n,m}(g; d\psi)$ *are the remainder terms in the corresponding Chakalov-Popoviciu formulae of Lobatto and Radau type.*

PROOF. We shall prove (5.18). As in [5] we have

$$f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(1)}{k!}(x-1)^k + \int_0^1 r_x(t)\, d\psi(t). \tag{5.20}$$

By (5.11)

$$s_{n,m}(x) = \sum_{k=0}^{m} \frac{p_m^{(k)}(1)}{k!}(x-1)^k + \sum_{\nu=1}^{n} \sum_{i=m-2s_\nu}^{m} \frac{m!}{i!} A^L_{m-i,\nu}(\tau_\nu - x)^i_+ \tag{5.21}$$

and changing indices $(k = m - i)$, the third sum on the right becomes

$$\sum_{i=m-2s_\nu}^{m} \frac{m!}{i!} A^L_{m-i,\nu}(\tau_\nu - x)^i_+ = \sum_{k=0}^{2s_\nu} \frac{m!}{(m-k)!} A^L_{k,\nu}(\tau_\nu - x)^{m-k}_+ = \sum_{k=0}^{2s_\nu} A^L_{k,\nu} r_x^{(k)}(\tau_\nu).$$

Equation (5.21) may be rewritten as

$$s_{n,m}(x) = \sum_{k=0}^{m} \frac{p_m^{(k)}(1)}{k!}(x-1)^k + \sum_{\nu=1}^{n} \sum_{k=0}^{2s_\nu} A^L_{k,\nu}\, r_x^{(k)}(\tau_\nu). \tag{5.22}$$

Subtracting (5.22) from (5.20) gives

$$f(x) - s_{n,m}(x) = \int_0^1 r_x(t)\, d\psi(t) + \sum_{k=0}^{m} \frac{1}{k!} \left( f^{(k)}(1) - p_m^{(k)}(1) \right) (x-1)^k \\ - \sum_{\nu=1}^{n} \sum_{k=0}^{2s_\nu} A_{k,\nu}^{L} r_x^{(k)}(\tau_\nu)$$

which, by virtue of (5.12) and (5.4), yields

$$f(x) - s_{n,m}(x) = \int_0^1 r_x(t)\, d\psi(t) - \sum_{k=0}^{m} \frac{m!}{k!} \beta_{m-k} (1-x)^k - \sum_{\nu=1}^{n} \sum_{k=0}^{2s_\nu} A_{k,\nu}^{L} r_x^{(k)}(\tau_\nu).$$

But

$$r_x^{(k)}(0) = 0, \quad r_x^{(k)}(1) = \frac{m!}{(m-k)!} (1-x)^{m-k}, \quad k = 0, \dots, m,$$

so that

$$f(x) - s_{n,m}(x) = \int_0^1 r_x(t)\, d\psi(t) - \sum_{k=0}^{m} \beta_{m-k} r_x^{(m-k)}(1) - \sum_{\nu=1}^{n} \sum_{k=0}^{2s_\nu} A_{k,\nu}^{L} r_x^{(k)}(\tau_\nu)$$

as claimed in (5.18).

The proof of (5.19) is entirely analogous to the proof of (5.18) and it shall be omitted.

## 6. On an analytic formula for the coefficients $A_{i,\nu}$ in (1.4)

Let

$$\omega_\nu(t) = \frac{\prod_{l=1}^{n} (t - \tau_l)^{2s_l+1}}{(t - \tau_\nu)^{2s_\nu+1}}.$$

On the basis of Hermite's interpolation (see [1, pp. 163–173]) we obtained the weights $A_{i,\nu}$ in the generalized Chakalov-Popoviciu quadrature formula (1.4) (see [15])

$$A_{i,\nu} = \frac{1}{i!} \sum_{k=0}^{2s_\nu - i} \frac{1}{k!} \left[ \frac{(t-\tau_\nu)^{2s_\nu+1}}{\Omega(t)} \right]_{t=\tau_\nu}^{(k)} \int_{\mathbb{R}} \frac{\Omega(t)}{(t-\tau_\nu)^{2s_\nu - i - k + 1}}\, d\lambda(t), \tag{6.1}$$

where

$$\Omega(t) = (t-\tau_1)^{2s_1+1} (t-\tau_2)^{2s_2+1} \cdots (t-\tau_n)^{2s_n+1} = \prod_{l=1}^{n} (t-\tau_l)^{2s_l+1},$$

and $i = 0, 1, \dots, 2s_\nu$, $\nu = 1, \dots, n$.

In the following statement we shall obtain an alternative expression.

LEMMA 6.1. *The coefficients $A_{i,\nu}$ in (1.4) can be expressed in the form*

$$A_{i,\nu} = \frac{1}{i!(2s_\nu - i)!}\left[\frac{1}{\omega_\nu(t)}\int_{\mathbb{R}}\frac{\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1} - \prod_{l=1}^{n}(t-\tau_l)^{2s_l+1}}{x-t}\,d\lambda(x)\right]_{t=\tau_\nu}^{(2s_\nu - i)}, \quad (6.2)$$

*where $i = 0, 1, \ldots, 2s_\nu$, $\nu = 1, \ldots, n$.*

PROOF. If we put $k = 2s_\nu - i - m$ in (6.1), then we have

$$A_{i,\nu} = \frac{1}{i!}\sum_{m=0}^{2s_\nu - i}\frac{1}{(2s_\nu - i - m)!}\left[\frac{(t-\tau_\nu)^{2s_\nu+1}}{\prod_{l=1}^{n}(t-\tau_l)^{2s_l+1}}\right]_{t=\tau_\nu}^{(2s_\nu - i - m)} \times \int_{\mathbb{R}}(x-\tau_\nu)^{2s_\nu - m}\prod_{\substack{l=1\\ l\neq\nu}}^{n}(x-\tau_l)^{2s_l+1}\,d\lambda(x).$$

Therefore

$$A_{i,\nu} = \frac{1}{i!}\sum_{k=0}^{2s_\nu - i}\frac{1}{(2s_\nu - i - k)!}\left[\frac{1}{\omega_\nu(t)}\right]_{t=\tau_\nu}^{(2s_\nu - i - k)}\int_{\mathbb{R}}(x-\tau_\nu)^{2s_\nu - k}\frac{\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}}{(x-\tau_\nu)^{2s_\nu+1}}\,d\lambda(x),$$

that is,

$$A_{i,\nu} = \frac{1}{i!(2s_\nu - i)!}\sum_{k=0}^{2s_\nu - i}\binom{2s_\nu - i}{k}\left[\frac{1}{\omega_\nu(t)}\right]_{t=\tau_\nu}^{(2s_\nu - i - k)} \times \int_{\mathbb{R}}\frac{(-1)^{k+1}k!\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}}{(\tau_\nu - x)^{k+1}}\,d\lambda(x). \quad (6.3)$$

For $p = 0, \ldots, k$, $k = 0, \ldots, 2s_\nu - i$, $i = 0, \ldots, 2s_\nu$, $\nu = 1, \ldots, n$, we have

$$\left[\prod_{l=1}^{n}(t-\tau_l)^{2s_l+1} - \prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}\right]_{t=\tau_\nu}^{(p)} = \begin{cases} -\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}; & p = 0,\\ \left[\prod_{l=1}^{n}(t-\tau_l)^{2s_l+1}\right]_{t=\tau_\nu}^{(p)}; & p > 0.\end{cases}$$

If $p > 0$, then by using the Leibniz formula we have

$$\left[\prod_{l=1}^{n}(t-\tau_l)^{2s_l+1}\right]_{t=\tau_\nu}^{(p)} = \left[(t-\tau_\nu)^{2s_\nu+1}\omega_\nu(t)\right]_{t=\tau_\nu}^{(p)} = \sum_{m=0}^{p}\binom{p}{m}\left[(t-\tau_\nu)^{2s_\nu+1}\right]_{t=\tau_\nu}^{(m)}\left[\omega_\nu(t)\right]_{t=\tau_\nu}^{(p-m)} = 0.$$

Therefore

$$\left[\prod_{l=1}^{n}(t-\tau_l)^{2s_l+1}-\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}\right]_{t=\tau_\nu}^{(p)}=\begin{cases}-\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}; & p=0,\\ 0; & p>0.\end{cases}$$

For the integral in (6.3) we have

$$\begin{aligned}
&\int_{\mathbb{R}}\frac{(-1)^{k+1}k!\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}}{(\tau_\nu-x)^{k+1}}\,d\lambda(x)\\
&\quad=\int_{\mathbb{R}}\frac{(-1)^k\cdot k!}{(\tau_\nu-x)^{k+1}}\left(-\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}\right)d\lambda(x)\\
&\quad=\binom{k}{0}\int_{\mathbb{R}}\left[(t-x)^{-1}\right]_{t=\tau_\nu}^{(k-0)}\left(-\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}\right)d\lambda(x)\\
&\qquad+\sum_{p=1}^{k}\binom{k}{p}\int_{\mathbb{R}}\left[(t-x)^{-1}\right]_{t=\tau_\nu}^{(k-p)}\left[\prod_{l=1}^{n}(t-\tau_l)^{2s_l+1}-\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}\right]_{t=\tau_\nu}^{(p)}d\lambda(x)\\
&\quad=\sum_{p=0}^{k}\binom{k}{p}\int_{\mathbb{R}}\left[(t-x)^{-1}\right]_{t=\tau_\nu}^{(k-p)}\left[\prod_{l=1}^{n}(t-\tau_l)^{2s_l+1}-\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}\right]_{t=\tau_\nu}^{(p)}d\lambda(x)\\
&\quad=\int_{\mathbb{R}}\left[\frac{\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}-\prod_{l=1}^{n}(t-\tau_l)^{2s_l+1}}{x-t}\right]_{t=\tau_\nu}^{(k)}d\lambda(x).
\end{aligned}$$

Now (6.3) becomes

$$\begin{aligned}
A_{i,\nu}=&\frac{1}{i!(2s_\nu-i)!}\sum_{k=0}^{2s_\nu-i}\binom{2s_\nu-i}{k}\left[\frac{1}{\omega_\nu(t)}\right]_{t=\tau_\nu}^{(2s_\nu-i-k)}\\
&\times\int_{\mathbb{R}}\left[\frac{\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}-\prod_{l=1}^{n}(t-\tau_l)^{2s_l+1}}{x-t}\right]_{t=\tau_\nu}^{(k)}d\lambda(x),
\end{aligned}$$

that is, (6.2) holds.

REMARK C. The formula (6.1) has been used for numerical calculation of the coefficients $A_{i,\nu}$ in (1.4) (see [15]). The expression (6.2) may be of interest for theoretical considerations. For example, the term

$$\int_{\mathbb{R}}\frac{\prod_{l=1}^{n}(x-\tau_l)^{2s_l+1}-\prod_{l=1}^{n}(t-\tau_l)^{2s_l+1}}{x-t}\,d\lambda(x)$$

is similar to the associated polynomials of the second kind (or the numerator polynomials) corresponding to the ordinary orthogonal polynomials (see [4, p. 86]). (In the case of $s_1=s_2=\cdots=s_n=0$ it is precisely that.)

**Acknowledgements.** This paper is dedicated to the memory of Professor Alessandro Ossicini. This work was supported in part by the Serbian Ministry of Science and Technology.

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