

# FIBONACCI - HIS RABBITS AND HIS NUMBERS and KEPLER

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## **PREAMBLE**

Leonardo of Pisa (of leaning tower fame), better known to us as Fibonacci, was **born** about **1175** and died about 1250. His fame rests mainly on his book **Liber Abacci** (A Book about the abacus or, the book of calculations - in fact its objective was to make the abacus obsolete) which he wrote in **1202**.

A second revised version was produced in 1228 and this great document translated with many errors by Boncompagni (see reference 2a) in 1857 is what has survived to this day and gives us an **almost complete statement of the mathematical knowledge that had accumulated in the West since the Greeks**. It can also be regarded as a comprehensive **merchant's handbook** as its stated aim was to demonstrate to these merchants the advantages of the **Hindu-Arabic** number system compared with the Roman system which apart from its clumsiness didn't even have a zero symbol. However the merchants then being much the same slaves to conformity as their present day counterparts rejected his system. It in fact took many decades to be accepted commercially; the transition was much more painful and time consuming than for example the recent conversion from British to metric measurements.

Unfortunately for posterity all that is known of him is contained in the second short paragraph of the second edition. Fortunately a numerate classics scholar Richard Grimm has recently taken an interest in his Book and studied the manuscript in Florence on which the Boncompagni was based (see reference 5a) and interesting work should emerge from that study. Leonards father was a merchant and customs officer and the family lived in Bugia (Morocco) during his childhood.

Thus he was exposed to Arabic influences at an early age and in particular he was familiar with the works of the great mathematician **al-Khwarizmi** dating from about 825. Born about 780 this mathematician and astronomer was in fact a Persian who lived in Baghdad and wrote two books which are known to us. One was on the Hindu-Arabic positional method and from

the title of the other (al-jabr) came the word **algebra**. A misspelling of his name gives us **algorithm** which is now one of the most important words in computer science.

When we say that Fibonacci's book was produced we mean that it was copied by hand as printing had not been invented. However even after printing had been established it was still not printed, incredibly, until 1857 see Boncompagni, reference [2]. And even now there is not an English translation! Regarding mathematical books we note the following:

1. The **very first printed book** devoted entirely to **mathematics** appears to be the "Treviso Arithmetic", named after Treviso, a small town near Venice. The book was printed in 1478 in the Venetian dialect which was one of the sources of modern Italian (the main source was of course Tuscan the language of Dante). Like the Liber it is mainly concerned with an explanation of the Hindu-Arabic system (see Deal [3], [9]).

2. It is a little known but significant fact that until recently, next to the Bible, **the book with the highest number of copies printed and the widest distribution was a mathematical book namely; Euclid's Elements.**

Fibonacci had several aliases which appear in his manuscripts. In Latin these are as follows:

*Leonardus Pisanus* (Leonard the Pisan).

*Leonardus filius Bonaccii*, (Leonard Fibonacci). Fibonacci was the family surname in Italian. and literally means "son of the simpleton (Bonaccio)"

*Leonardus Bigollus* "Bigollo" in Tuscan dialect is roughly translated as "absent minded" or even "**blockhead**", and this nickname probably arises out of his family name. (It also meant traveller).

From this it is not surprising that he was called "the dunce", son of "the dope" even though his father's name was in fact "Guilielmus" (William) and not as commonly believed Bonaccio. It was not until 1838 that ". . . *the sobriquet Fibonacci was foisted on Leonardo by the mathematical historian Libri.*" (see reference 2b).

However today he is remembered in general only through his **Fibonacci numbers which arise out of the rabbit problem**. This is unfortunate as although the mathematics associated with Fibonacci Numbers is widespread, deep, and full of mysterious patterns he most certainly did not carry out any analysis of these numbers apart from establishing that each term is equal to the sum of the preceding two and mentioning that the process goes on indefinitely. The numbers were not even referred to as Fibonacci until 1877.

His major achievements are at a far higher level than this and are detailed in Gies, references [4] and [2b]. Of special importance is his work on Diophantine analysis (Pythagorean triplets)

and congruent numbers. His method for determining numbers that can be added to squares to obtain squares is probably one of the most important results in number theory before Fermat. In particular he analysed the cubic

$$x^3 + 2x^2 + 10x = 20,$$

and showed that it cannot be solved by square roots. As well he detailed an extremely accurate method for solving this equation numerically (having shown that it cannot be in the form  $\sqrt{a + \sqrt{b}}$  and hence is not associated with a geometric construction using straight edge and compass). It should be emphasised that algebraic methods at this stage did not use any special notation; that is, they were rhetorical; it took almost 500 years to develop the symbolic notation that we now use. Such notation is convenient but as that Prince of mathematicians Gauss said, what mathematicians need are “notions not notations”.

**His book begins** *The nine Indian figures are : 987654321. With these figures, and with the sign 0. . any number may be written, as is demonstrated below.*

It then goes on for seven chapters to describe these new numerals and show how they may be applied to practical problems. It should be pointed out that the Arabs were using the base 60- the Babylonian sexagesimal, from which we inherited our angular units of minutes and seconds. So his book contains both base 10 and 60. However, he did **not use decimal fractions**, even though they had been developed by the Arabs about 952. Simon Stevin (see references [5] and [8]) appears to be the first westerner to have systematically developed the use of such fractions but their use did not become wide spread in the western world until the end of the sixteenth century.

Fibonacci had a rather strange way of dealing with fractions which he represented as sums of reciprocals, so that  $5/6$  becomes  $1/3 + 1/2$ . These are called **Egyptian fractions** (see reference 5b). It is interesting to imagine what might have developed if he had been lead into continued fractions by such a device.

No biography or painting is known of him but this is not unusual even for Kings and Bishops in the middle ages. But one thing is certain from his writings. Even though he dedicated his book to a famous astrologer and lived at a time when numerology, astrology and quakery were very fashionable, he was a very down to earth mathematician; the complete contrast to the Pythagorean mystics who now seem to dominate the ranks of those studying in this area.

It is of interest to note that the next record of the Fibonacci numbers being studied is in the work of **Kepler**, in his small monograph (1611), just on four centuries after Fibonacci, “Strena seu de Nive Sexangula (**A New Year Gift : On Hexagonal Snow** perhaps a better translation is - on the six cornered snowflake, see reference [7] and the annotation). Kepler was apparently quite unaware of Fibonacci’s work and as well as rediscovering the sequence and the recurrence property he made a remarkable discovery about the ratios of consecutive terms of the Fibonacci sequence

1, 1, 2, 3, 5, 8, . . . “ . . . as 5 is to 8 so is 8 to 13, practically, and as 8 is to 13, so is 13 to 21 almost” and guesses that these ratios approximate what he calls the divine proportion and what we call the golden section (defined below). That is he saw that the ratios  $1/1, 2/1, 3/2, 5/3, . . .$  rapidly approach  $\phi = 1.618$ . In fact the first ratio is below, the second above, the third below and so on. He then goes on to say that he believes that the “*seminal faculty*” (that is the reproductive process) “*is developed in a way analogous to this proportion which **perpetuates** itself.*” This is a particularly insightful comment as he is implying that Fibonacci numbers and hence  $\phi$  are fundamental to the biological process of **self replication**. The insight was overlooked by biologists until comparatively recently only being picked up with the serious study of phyllotaxis - leaf arrangements in plants, in the nineteenth century.

His little book is mainly concerned with the packing together of circles in a plane and spheres in space. It also describes the structure of flowers in quincuncial arrangements (arrangements of 5 things placed at each of the four corners of a square) and this moves Kepler to the belief that they express “*an emanation of a sense of form, and feeling for beauty, from the soul of the plant*”.

Also Kepler examines the regular Platonic solids and in particular the dodecagon and icosahedron and guesses that “*the structure of these solids in a form so strikingly pentagonal could not come to pass apart from that proportion which geometers today pronounce divine*”.

From this it is seen that unlike Fibonacci, Kepler was very much a mystic so much so that he insisted, and even doctored some of his data to show, that the planetary orbits were still based on the Platonic solids, even after discovering the more elegant model where the orbits are ellipses with the sun at a focus. In doing this he was the first to demonstrate that after 2000 years of lying waiting, conic sections could be applied to a real world problem. That’s right! 2000 years previously the mathematics of these conic sections (parabolas, ellipses, and hyperbolas formed by cutting a cone with a plane) were first studied exhaustively (in an eight volume masterpiece which was the standard reference on conics until recent times ) by the Greek **Apollonius** of Perga who, next to Archimedes, Pappas and Pythagoras, was probably the most important mathematician of antiquity. He devoted the whole of his life to the study of these conics and never even considered that there could be any real world applications.

There are many other examples of important mathematics being developed initially only for purity and then turning out to be vital in some practical application. Perhaps the most dramatic of these in recent times occurs with the pure development of non-Euclidean geometry (originally begun by Gauss, thence, by Riemann and perfected by Minkowski) which was so vital for the conception and formulation of Einstein’s Theory of Relativity.

### **The Golden Section - Aureo Section**

The Golden section was an object of great interest during two of the golden ages of history:

firstly with the Greeks and then during the European Renaissance. Was this a coincidence or was this object a symbol of the freedom and vitality of thought that is associated with a resurgence of creativity? Let us hope that the latter is correct because it has again reemerged as an object of some interest to both biologists and mathematicians in such apparently diverse areas as optimisation, computing science and pattern formation.

Aureo secto is the L. literal translation of golden mean. Allied terms are: **aurea mediocritas** which sometimes means “the happy mean” (not to be confused with a sun tanned ocker); **ariston metron** which also means “the middle course is best” (course comes from ariston = the Greek for breakfast from which we get aristology = the art of dining).

We will use the symbol  $\phi$  for this Golden section and define it as the positive solution to the quadratic equation  $x^2 - x - 1 = 0$ . Hence  $\phi = (1 + \sqrt{5})/2 = 1.618033989..$  (noting that as  $\phi$  involves the square root of 5 it is thus an algebraic irrational, that is, it is the solution of a polynomial equation. In contrast a number like  $\pi$  although irrational, cannot be expressed as the root of a polynomial equation and is called a transcendental number). The symbol  $\phi$  is appropriate as it the first letter of **Phidias** the name of the Greek sculptor who often used the Golden section in proportioning his sculptures. If one frames the Pantheon with a rectangle it is found to have sides in the ratio of  $\phi$ . However mathematicians today more commonly represent this number by  $\tau$ , tau, which is the first letter of the Greek word for section or cut. (May we be excused for now referring to Fibonacci as  $\phi$ -Bonacci).

The first recorded awareness of  $\phi$  appears to be with the Pythagoreans towards the latter half of the 6th century BC who knew of the various Golden relationships within the pentagon. The first written record we have is in Euclid’s elements (about 3rd century BC) where the following problem is solved. “To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segments” (Book II Propostion 11). “To cut a given finite line in extreme and mean ratio” (Book IV, 11). It appears that the first technique is due to the Pythagoreans and the second is due to Euclid. From these techniques we are led into the construction of a golden triangle (an isosceles triangle whose base angles are each twice the other angle) (IV, 10) and hence to the construction of the regular pentagon (IV, 11). It was not until 1844 that the term “Golden section” was first introduced. Previous to that the term “Divine proportion” was generally used. In 1509 **Fra Pacioli** had his famous book printed “**Divina Proportion**” which listed almost as objects of veneration many of the properties of  $\phi$ . The geometrical drawings for this book were done by Leonardo da Vinci. It is now known that Pacioli copied much of his book from a manuscript of **Pier della Francesca** which is now in the Vatican. The terms “sectio divina” and “proportio divina” are both found in the writings of Kepler.

From the quadratic equation

$$x = 1 + 1/x.$$

Iterating we have  $x = 1 + 1/(1 + 1/x)$

from which  $\phi = 1 + 1/(1 + 1/(1 + 1/(1 + \dots)))$ .

This expression is called a continued fraction, from which it can be shown that, not only is  $\phi$  irrational but that it is the most difficult of all numbers to approximate by a rational fraction. Because of this infamy it can be regarded as the most irrational of the irrational numbers.

### Definitions and Properties

So for starters we have the **Fibonacci sequence** (which was first given this name in 1877)

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, . . . (**F**)

We generate this by writing down the first two terms 1, 1 and then using the following rule (algorithm) to generate the next term. "Form the next term by adding the last two terms in the sequence". We will simply call this sequence **F** from now on. Let us introduce some terminology. The symbol  $u_j$  for the  $j$ th term is sometimes used but we prefer the symbol  $F_j$  to remind ourselves that we are dealing with this sequence rather than one of the allied sequences. Thus if we call the first term  $F_1$  and so on, then we have the recurrence

$$F_1 = F_2 = 1,$$

$$F_{j+2} = F_{j+1} + F_j, \text{ for } j = 1, 2, 3, 4, \dots (\mathbf{R})$$

This results in  $F_5 = 5$ , which is a good check that you have written down the sequence correctly. Sometimes  $F_0$  is defined as zero in which case we start by defining  $F_1 = 1$  and then (R) holds for  $j = 0, 1, 2, \dots$

The following **computer program** outputs the terms  $j = 1$

$F = 1$ : " $F$  is  $j$ th term  $P$  is  $(j - 1)$ th term"

$P = 0$

print  $F, P$ : "at this stage we have  $F_1 = 1, P_1 = 0$ "

Label  $j = j + 1$

$X = F$

$F = F + P$

$Phi = F/P$ : "This will approach golden section"

Print  $Phi$

$P = X$

Print  $F, j$ : This gives  $F_j$  starting with  $F_2$

IF  $j$  is big enough stop, otherwise go to Label.

We can readily check on our calculator that Kepler's observation holds and that the ratio of successive terms does indeed tend towards  $\phi$ . This can be shown as follows

$$\begin{aligned} F_{i+1}/F_i &= (F_i + F_{i-1})/F_i = 1 + 1/(F_i/F_{i-1}) \\ &= 1 + 1/(1 + 1/(F_{i-1}/F_{i-2})) = 1 + 1/(1 + 1/(1 + \dots)) = \phi. \end{aligned}$$

## SOME PROPERTIES

In terms of beautiful patterns F is one of the richest sequences that can be imagined. Here are some of the patterns which can be readily checked especially if you have access to a computer.

**P1.** Every third term is divisible by 2, every fourth term is divisible by 3, every fifth term is divisible by 5, every sixth term is divisible by 8 and so on. In other words  $F_n$  divides the  $n$ th term, as well as the  $2_n$  th,  $3_n$  th,  $4_n$  th, terms etc.

**Hence  $F_n$  divides every term  $F_{nk}$  for  $k = 1, 2, \dots$**

In particular we note that we always have two odd terms separated by an even term, three terms not divisible by 3 separating a pair of terms that is, and so.

**P2.** The last digits in each term repeat a cycle of 60 numbers and the last two digits repeat a cycle of 300 numbers and so on (the repeating cycle is 1,500 for the last three digits, 15,000 for the last four digits, 150,000 for the last five and so on).

**P3.** The square of each term differs by 1 from the product of the two terms on either side. This difference is alternatively plus or minus as we progress. (This result was proved by J.D. Cassini in 1680 (Histoire Acad. Roy. Paris 1, 201, although Simpson proved it independently in 1753, [9]).

$$F_{n+1}F_{n-1} - (F_n)^2 = (-1)^n.$$

That is

$$\frac{F_{n+1}}{F_n} - \frac{F_n}{F_{n-1}} = \frac{(-1)^1}{F_n F_{n-1}}$$

That is the difference between two successive ratios of Fibonacci numbers is alternatively plus or minus. Furthermore, this difference gets progressively smaller confirming that the ratio does converge to  $\phi$  in the limit.

**P4. Consecutive terms are relatively prime** (that is they cannot have a common factor other than 1). This follows from the expression in 3. If  $F_{n+1}$  and  $F_n$  had a common factor it would also have to be a factor of  $(-1)^n$  which is impossible if it were other than unity.

**P5.** The sum of the squares of any two consecutive terms is another term which we note

has an odd subscript (first proved by Lucas in 1876)

$$(F_n)^2 + (F_{n+1})^2 = F_{2n+1}.$$

Now form a sequence made up of the squares of the terms then, from this form a new sequence made up of the sums of consecutive squares. It follows from our relationship that this new sequence will be the same as the sequence of terms with odd subscripts.

**P6.** For any **four consecutive terms A,B,C,D** :  $C^2 - B^2 = AD$ .

**P7.** With the exception of 3, **every term that is prime has a prime subscript** (thus 233 is prime and has a prime subscript 13). However be warned that the converse does not hold: a prime subscript does not mean that the term is prime. Our first counter example does not occur until  $F_{19} = 4,181$  where although 19 is prime  $4,181 = 37$  times 113 and is thus composite.

In fact if the converse held it would tell us that the number of terms which are prime is infinite. **As it is we just do not know how many terms are prime.** That is we just do not know whether a largest prime Fibonacci number exists. This is one of the many unsolved Fibonacci problems.

**P8.** It was only recently proved that  $F_{12} = 144$ , **is the only term (apart from the trivial case of 1) which is a square.** Surprisingly it also happens to be the square of its subscript.

**P9.** For every integer  $i$  there are an infinite number of terms that can be **divided by i** and at least one can be found amongst the first  $i^2$  terms.

**P10.** The sum of the first  $n$  terms is one less than a Fibonacci number. In fact the **sum of the first n terms is  $F_{n+2} - 1$ .** (Again this result is due to Lucas).

**P11.** If we expand  $G(z) = 1/(1 - z - z^2)$  as a power series in  $z$  we obtain

$$F_1 + F_2z + F_3z^2 + F_4z^3 + \dots + F_i z^{i-1} + \dots$$

That is the coefficient of  $z^{i-1}$  generates the  $i$ th Fibonacci number and hence **G(z) is the generating function for the Fibonacci numbers.**

This is readily checked. Multiply the power series succesively by  $1, -z,$  and  $-z^2,$  regroup the terms and everything else cancels out leaving 1.

**P12.** We note that the reciprocal of  $F_{11} = 89,$  can be written in the following curious way. Start with .0 then add the first term obtaining .01 then add the next term obtaining .011 and so on from which

.0112358



$$\begin{array}{r}
13 \\
21 \\
34 \\
55 \\
89 \\
\dots \\
\hline
.0112359550. = 1/89
\end{array}$$

Also  $10000/9899 = 1.010203050813213455. . .$ , which follows immediately from substituting  $z = 0.01$  in **11**.

**P13.** As

$$\begin{aligned}
F_j &= F_{j-1} + F_{j-2} \\
F_j/F_{j-1} &= 1 + 1/F_{j-1}/F_{j-2} \\
&= 1 + 1/(1 + 1/F_{j-2}/F_{j-3}) \\
&= 1 + 1/(1 + 1/(1 + 1/(1 + \dots
\end{aligned}$$

which is in the form of a continued fraction. As  $j$  becomes unbounded this approaches the golden section  $\phi = 1.628. .$

**P14.** The number of kilometers in a mile (1.609..) is roughly equal to  $\phi$ , hence we can do conversions as follows:

a) if we have a speed of  $F_j$  mph then this is about  $F_{j+1}$  kilometers per hour

b) otherwise we express the kph say  $x$  in terms of a sum of Fibonacci numbers and then reduce each term to the next smaller Fibonacci number - (this is roughly equal to dividing by  $\phi$ ). The simplest way to get our sum is to subtract the largest Fibonacci number less than  $x$  then repeat for the remainder etc.

Thus  $60 \text{ kmh} = 55 + 5$  gives  $34 + 3 = 37 \text{ mph}$

whereas going up in similar fashion

$60 \text{ mph} = 55 + 5$  gives  $84 + 8 = 92 \text{ kmh}$ .

These Fibonacci patterns are so rich and intertwined that we could literally fill books just describing them, and shelves of books proving our results. In fact a journal was started in 1963, devoted exclusively to reporting such patterns and their impact on number theory, namely The Fibonacci Quarterly (see reference [1]). Also see Vorob'ev references [11], [12] for an elementary mathematical introduction and more recently Steven Vajda [15].

### Some Important Relationships

Probably the most important relationship that comes out of our sequence is what is sometimes called the **Binet formula** (after the French mathematician (1786 to 1856) which

relates Fibonacci numbers to the golden section  $\phi$  as follows

$$\sqrt{5}F_j = \phi^j - (-1/\phi)^j, \quad (\text{B})$$

or

$$\sqrt{5}F_j = \{(1 + \sqrt{5})/2\}^j - \{(1 - \sqrt{5})/2\}^j$$

or

$$F_j = (\phi^j - y^j)/(\phi - y),$$

where

$$y = -1/\phi = (1 - \sqrt{5})/2,$$

and thus  $\phi$  is the positive root and  $y$  the negative root of

$$x^2 - x - 1 = 0.$$

We also note that

$$\phi y = -1 \text{ and that } \phi - y = \phi + 1/\phi = \sqrt{5}.$$

Firstly we should point out that in fact this formula was **first discovered and proved by Daniel Bernoulli in 1724**. In 1726 Euler mentioned the formula in a letter to Bernoulli. Shortly afterwards A. de Moivre ("Miscellanea Analytica - London 1730 obtained the result in the first systematic account of linear recurrences, pages 26 to 42). Although Binet appears to have discovered the formula independently he did not publish his findings until 1843. The formula is often mistakenly attributed to Lucas who may have rediscovered it but he did not report it until 1876.

(B) tells us a great deal and is used as the basis for the proof of all sorts of other relationships. As the second term decreases in absolute magnitude with  $j$  we can immediately state that for large  $j$ ,  $F_j$  approaches (is asymptotic to)  $\theta^j/\sqrt{5}$ . In fact, as is readily checked, this is quite a good approximation even for values of  $j$  as low as 5 (when the error is about 4%). An **easier** way to carry out the calculations is to simply calculate  $\theta^j/\sqrt{5}$  and take  $F_j$  as the **nearest integer**. This is equivalent to

$$F_j = [\phi^j/\sqrt{5} + 1/2],$$

where  $[x]$  is the integer part of  $x$ .

We now offer a **proof of (B)**. To do this we show immediately by substitution that  $B$  satisfies  $R$ . Then we show that there is no other solution. Suppose to the contrary that there is another solution  $u_j$ . Then  $z_j$  would also satisfy (B) where  $z_j = F_j - u_j$ , with  $z_0 = z_1$ . Induction shows immediately that  $z_j = 0$  for all  $j$  and hence we have proved that  $B$  is the unique solution to  $R$ .

An **alternative** derivation of this formula is to note that the generating function can be expressed in partial fractions as follows

$$\begin{aligned}\sqrt{5}G(z) &= 1/(\phi - z) + \phi/(1 - \phi z). \\ &= -y(1/(1 + zy) + \phi(1/(1 - \phi z))).\end{aligned}$$

Now expand each of the terms in brackets as a power series obtaining

$$\begin{aligned}\sqrt{5}G(z) &= \phi(1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \dots + \phi^j z^j + \dots) \\ &\quad -y - y^2 z - y^3 z^2 - y^4 z^3 + \dots - y^{j+1} z^j + \dots.\end{aligned}$$

But from 11, the coefficient of  $z^j$  must be  $F_{j+1}$  hence (B) is true.

Another powerful result is

$$\phi^j = F_j \phi + F_{j-1}, \text{ for } j = 2, 3, \tag{C}$$

**Proof** By induction: we firstly note that it is true for

$$j = 2, \text{ as } \phi^2 = \phi + 1, \text{ and } F_2 = F_1 = 1.$$

Next we assume that it is true for  $j = k$  that is that  $\phi^k = F_k \phi + F_{k-1}$ .

But assuming this

$$\begin{aligned}\phi^{k+1} &= \phi^k \phi = F_k \phi^2 + F_{k-1} \phi = F_k(1 + \phi) + F_{k-1} \phi \\ &= (F_k + F_{k-1})\phi + F_k = F_{k+1} \phi + F_k,\end{aligned}$$

and hence the result is true for  $j = k + 1$  as required. We also note that

$$\phi^j = (F_j \sqrt{5} + u)/2, \text{ where } u = F_{j-1} + F_{j+1}.$$

This follows when we recall that  $\phi + 1/\phi = \sqrt{5}$ .

It should be emphasised that this expression allows us to make  $j$  the subject of our relationship and thus we can obtain the exponent of  $\phi$  given any two consecutive Fibonacci numbers. That is

$$j = \log_{\phi}(F_j \sqrt{5} + u)/2.$$

Now using formula (C) as a basis we can offer a **very elegant derivation of (B)**. To do this we firstly note that  $y = -1/\phi$  also satisfies the basic recurrence (C), that is  $y^j = F_j y + F_{j-1}$  and if we subtract this from (C) we immediately obtain (B).

Perhaps the most powerful relationship from which many other identities may be established is the following formula of J. Halton [6].

$$F_m^k F_n = (-1)^{kr} \Sigma^k C_h (-1)^h F_r^h F_{r+m}^{k-h} F_{n+kr+hm},$$

with  $h$  ranging from 0 to  $k$ .

By substituting various values for  $k$ ,  $r$ ,  $m$  and  $n$  Halton was able to immediately establish **95 different Fibonacci relationships!**

### THE FIBONACCI RABBITS

This famous problem was first presented to the world in 1202 where we find on pages 123-128 of the manuscript of Fibonacci's **Liber Abacci**

*“Someone placed a pair of rabbits in a certain place, enclosed on all sides by a wall, so as to find out how many pairs of rabbits will be born there in the course of one year, it being assumed that every month a pair of rabbits produces another pair, and that rabbits begin to bear young two months after their own birth.*

*As the first pair produces issue in the first month, in this month there will be 2 pairs. Of these, one pair, namely the first one, gives birth in the following month, so that in the second month there will be 3 pairs. Of these, 2 pairs will produce issue in the following month, so that in the third month 2 more pairs of rabbits will be born, and the number of pairs of rabbits in that month will reach 5; of which 3 pairs will produce issue in the fourth month, so that the number of pairs of rabbits will then reach 8. Of these, 5 pairs will produce a further 5 pairs, which, added to the 8 pairs, will give 13 pairs in the fifth month. Of these, 5 pairs do not produce issue in that month but the other 8 do, so that in the sixth month 21 pairs result. Adding the 13 pairs that will be born in the seventh month, 34 pairs are obtained: added to the 21 pairs born in the eighth month it becomes 55 pairs in that month: this, added to the 34 pairs born in the ninth month, becomes 89 pairs: and increased again by 55 pairs which are born in the tenth month, makes 144 pairs in that month. Adding the 89 further pairs which are born in the eleventh month, we get 233 pairs, to which we add, lastly, the 144 pairs born in the final month. We thus obtain 377 pairs: this is the number of pairs procreated from the first pair by the end of one year.*

*A pair  
1  
First (Month)  
2  
Second  
3  
Third  
5  
Fourth  
8  
Fifth  
13  
Sixth*

21  
Seventh  
34  
Fig.1

*From [Fig.1]\* we see how we arrive at it: we add to the first number the second one i.e. 1 and 2; the second one to the third; the third to the fourth; the fourth to the fifth; and in this way, one after another, until we add together the tenth and the eleventh numbers (i.e. 144 and 233) and obtain the total number of rabbits (i.e. 377); and it is possible to do this in this order for an infinite number of months.”*

\*Fibonacci does all calculation tables and diagrams in his margin.

Now there are two ways of looking at this problem:

### **The cage model**

Here we consider a population made up of breeding pairs (a male and a female rabbit born at exactly the same moment). Each pair is kept in a **separate cage**. Initially we place a single pair of new born rabbits in a cage. For the first month they are infertile and unable to mate. After exactly one month from birth each rabbit becomes fertile and the male rabbit immediately impregnates the female. Exactly one month later (that is two months from birth) the female produces a litter of which a single mating pair survives with identical breeding habits to the parents. This newborn pair is placed in a separate cage and the process repeats itself. The rabbits may be regarded as immortal. In fact Fibonacci merely required each rabbit to be replaced at death or when it failed to breed.

Hence in this model we consider the number of **pairs** of rabbits (or the **number of cages**) as the population. This is why we call this **the cage model**.

### **The harem model**

We now consider the population made up entirely of **female** rabbits (a harem) with the following fecundity (fertility) and mortality characteristics. Each newborn female rabbit takes one month to mature at which stage it is impregnated by a male rabbit brought in from outside of the harem and not included in the population of the harem. It then takes exactly one more month to bring forth a litter which consists of a single female rabbit. Again it is immediately impregnated by the male from outside and produces another litter one month later. It follows that once mature, a female produces a single newborn each and every month. Each newborn rabbit then continues with the same breeding pattern as the mother. Again the rabbits may be regarded as immortal. This is why we call this **the harem model** and emphasise that we count the individual females in the population. In the model that follows we do not predict when the male rabbit will collapse through exhaustion.

So that we can refer to either model we will simply talk about a **unit**; in the cage model

this is a breeding pair in the harem model it is a single female.

We look at our population of units at the **very beginning of the  $n$ th month** and assume that births have just taken place in the preceding instant. The population will remain in this state for the next month until immediately after the next birth time in month  $n + 1$ .

Let us agree to count our population in the instant immediately **after** the birthtime in month  $n$  and furthermore agree to call this instant **time  $n$** .

We now classify our units into two types: **new** for newborn and **old** (mature or fertile). Hence we define  $N_n$  as the number of new **units**,  $O_n$  the number of old units, and  $T_n$  the total number of units; all measured at time  $n$  (that is immediately after the birth time of month  $n$ ). Immediately we have

$$T_n = O_n + N_n. \quad (1)$$

Suppose we start off our population with a **single old unit** (strictly speaking it should be 1 month old as we do not include any newborn) in the first month: that is at time one ( $n = 1$ ). Then at time 2 we would still have this unit as well as a single new unit. At time 3 we would have 3 units; the original unit, another 1 month old unit and a newborn unit. In terms of our new notation

$$T_1 = O_1 = 1, N_1 = 0 ; O_2 = N_2 = 1, T_2 = 2 ; O_3 = 2, N_3 = 1, T_3 = 3.$$

Now we try to discover some transition rules. Firstly we note that the number of new units at time  $n$  is the same as the number of old units at  $n - 1$ : that is each old unit one month ago will give birth now, even though one month ago it may only have just become mature. In the harem model this is equivalent to assuming that the outside male impregnated each and every one of the old females immediately after the birth time one month ago.

Hence

$$N_n = O_{n-1} \quad (2)$$

Also the old at time  $n - 1$  are still old at  $n$  and the new at  $n - 1$  become old at  $n$  and thus it follows that

$$O_n = O_{n-1} + N_{n-1} = T_{n-1}. \quad (3)$$

(3) into (2) gives

$$N_n = O_{n-1} = T_{n-2}. \quad (4)$$

We have now shown that: the new are equal to the old in the previous month; the old are equal to the total in the previous month; the new are equal to the total two months ago.

Substituting (2) into (3) we have

$$O_n = O_{n-1} + O_{n-2}, \quad (5)$$

where  $n \geq 3$  and  $O_1 = O_2 = 1$ .

This is the **Fibonacci recurrence** and thus with these initial conditions we have the sequence 1, 1, 2, 3, 5, . . . for old units, which is what is usually regarded as **the** Fibonacci sequence. That is

$$O_n = F_n. \quad (6)$$

Adding (2) and (3) we also obtain the Fibonacci recurrence for the total units, from which, with the above initial conditions we obtain

$$T_{n+1} = T_n + T_{n-1}, \quad (7)$$

where  $n \geq 3$  and  $T_1 = 1, T_2 = 2$ .

From this we generate the sequence 1, 2, 3, 5, . . . . Hence

$$T_n = F_{n+1}. \quad (8)$$

The sequence for the new follows immediately from these two

$$N_{n+1} = N_n + N_{n-1}, \quad (9)$$

where  $n \geq 2$  and  $N_1 = 0, N_2 = 1$ . Hence the new sequence is 0, 1, 1, 2, 3, 5, . . . from which  $N_n = F_{n-1}$  with  $F_0 = 0$ .

On the other hand if we **started** off the population with a **new unit** instead of an old unit we would still have the same basic recurrences (5), (7), (9) with the slightly modified initial conditions  $O_1 = 0, O_2 = 1, T_1 = 1, T_2 = 2$ , from which  $N_1 = 1, N_2 = 0$ . Consequently our modified sequences are the original sequences with an extra first term added in each case as follows. The old becomes 0, 1, 1, 2, 3, 5, . . . and  $O_n = F_{n-1}$ . The total becomes 1, 1, 2, 3, 5, . . . and  $T_n = F_n$ . And the new becomes 1, 0, 1, 1, 2, 3, 5, . . . from which  $N_n = F_{n-2}$  with  $F_{-1} = 1$ . Although the sequence for the new starts off in a slightly different way from the standard Fibonacci it soon settles down to this pattern.

We summarise the above relationships in the following:

**TABLE 1**

**RABBIT POPULATION AGAINST TIME (n)**

OLD = start with 1 old, NEW = start with 1 new

Time (h)	OLD				NEW				Fn
	N	O	T	Total New Born upto n	Time (n)	N	O	T	
1	0	1	1	0	1	1	0	1	1
2	1	1	2	1	2	0	1	1	1
3	1	2	3	2	3	1	1	2	2
4	2	3	5	4	4	1	2	3	3
5	3	5	8	7	5	2	3	5	5
6	5	8	13	12	6	3	5	8	8
7	8	13	21	20	7	5	8	13	13
8	13	21	34	33	8	8	13	21	21
9	21	34	55	54	9	13	21	34	34
10	34	55	89	88	10	21	34	55	55
11	55	89	144	143	11	34	55	89	89
12	89	144	233	232	12	55	89	144	144

Let us now return to our model where we start with one old unit and do some more accounting. This time we count the total number of new units (that is newborn) up to and including time  $n$ , then this total plus the single initial old unit must be equal to the total number of units at time  $n$ .

$$1 + \sum_{j=1}^n N_j = T_n = F_{n+1}$$

But as we have  $F_0 = 0$ ,

$$1 + \sum_{j=2}^n F_{j-1} = 1 + \sum_{i=1}^{n-1} F_i = F_{n+1}$$

This confirms P10 which states that the first  $k$  Fibonacci numbers is one less than  $F_{k+2}$ .

**The Numbers in Each Generation**

The following is based on the article by Rose reference [7]. Suppose we define  $S(n, k)$  as the number of  $k$ th generation units at **time**  $n$ . By this we mean the following. Let us **start** our population with a single new unit at time  $n = 1$ . We will call this the **zeroth generation** and this generation is never added to but ages continuously. Any direct offspring from this unit we will call first generation and note that they will be produced at times 3, 4, 5 . . . Any direct offspring from these first generation old units will be called second generation and produced at times 5, 6, 7, . . . and so on. Hence  $S(1, 0) = 1$ . At time 2 there is no change, at time 3 the first birth takes place and it follows that  $S(3, 0) = S(3, 1) = 1$ , that is one first generation and our original zeroth generation. At time 4,  $S(4, 0) = 1, S(4, 1) = 2$ . At time 5 we have



our first second second generation unit and hence  $S(5, 0) = 1, S(5, 1) = 3, S(5, 2) = 1$ . From such considerations we derive the following **Transition rules** - For  $k$  greater than zero the number of  $k$ th generation units at the end of the  $n$ th month may be considered as made up of two components: those units that were  $k$ th generation in month  $n - 1$  and those units that were born from a  $k - 1$ th generation parent. The first component is simply  $S(n - 1, k)$ . The second component is  $S(n - 2, k - 1)$  as only the contribution from the  $k - 1$ th generation alive 2 months ago will give birth to  $k$ th generation progeny now at time  $n$ . Consequently we have established the recurrence relationship

$$S(n, k) = S(n - 1, k) + S(n - 2, k - 1), k > 0, n > 2, \quad (10)$$

$$S(n, 0) = 1, n \geq 1.$$

We note that time,  $n$  is what we previously called the new time and hence the subscript of the corresponding Fibonacci number for the sum of terms. Thus with the above recurrence and the **extra initial condition**,  $S(2, 1) = 0$ , we are able to generate **Table 2**. For example

$$S(8, 2) = S(7, 2) + S(6, 1),$$

or  $10 = 6 + 4$ . In terms of this table we look at the third entry corresponding to  $k = 2$  and see that 10 is equal to the sum of the term above it in this column plus the term in the second column in the next row above. We now show that

$$S(n, k) = {}^{n-k-1}C_k, \quad (11)$$

satisfies (10).

A well known Pascal relationship is

$${}^{n-k-1}C_k = {}^{n-k-2}C_k + {}^{n-k-2}C_{k-1}.$$

Substituting for  $S$  in this formula it is seen that we have reproduced (10). Also  $n = 2$  and  $k = 0$  and 1 in (11) satisfies our initial conditions. Consequently (11) satisfies the recurrence (10).

## TABLE 2 GENERATIONS

$n = TIME(new)$	$S(n, k)$
1	1
2	1, 0
3	1, 1
4	1, 2
5	1, 3, 1
6	1, 4, 3
7	1, 5, 6, 1
8	1, 6, 10, 4
9	1, 7, 15, 10, 1
10	1, 8, 21, 20, 5
11	1, 9, 28, 35, 15, 1
12	1, 10, 36, 56, 35, 6
13	1, 11, 45, 84, 70, 21, 1

There is another very interesting pattern in this table. For example at time 10 the numbers in the generations form the sequence 1, 8, 21, 20, 5. Now if we refer to the Pascal triangle we will see that this is the sequence of entries along the diagonal rising from the unit in the first (zeroth) column in the ninth row (counting the first row as zero). This is in accordance with the result

$$\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} n-k-1 C_k = F_n.$$

In effect what we have shown from this analysis is that, for example referring to the harem model, after time 10 months there is the original rabbit, there are 8 daughters of this original rabbit, there are 21 granddaughters, 20 great granddaughters and finally 5 great great granddaughters. However each generation except the zeroeth is spread over a wide range of ages and many of the great great granddaughters of this original rabbit are in fact older than some of her daughters.

### Age Groups

If we ask the question - how many units are of a particular age ? The answer is very simple. Because there are no deaths all units born  $j$  months ago, at time  $n - j$ , are now, at time  $n$ , of age  $j$  months. Thus if the number in this age group is called  $a_j$ , then

$$a_j = N_{n-j}.$$

In the model where we start with one old unit this becomes

$$a_j = N_{n-j} = F_{n-j-1},$$

Another way of looking at our Fibonacci Rabbits is to regard the lifetime of each unit as one month. For this model we again classify our units into new and old but: an old unit after one month dies and is replaced by two units, one old and one new; a new unit after one month dies and is replaced by an old unit.

This model has a certain convenience when representing the problem as a tree or a branching process.

### **Average Age in Months**

We now ask the question - what is the average age of the rabbits? Let us be more specific - assuming that we started with one pair one month old in 1202, when the Fibonacci book was released, what is the average age of the population now. Did you think that this figure would be very large - well you are wrong - it can be shown (Fib. Q. V.26 page 418) that the average age very quickly tends towards, well of course, the Golden section!

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**NOTE:** Apart from the above, the Fibonacci Association have sponsored a series of **international conferences on Fibonacci numbers and their applications** beginning with a conference in Patras in **Greece in 1984** with a second conference in **San Jose in 1986**. The proceedings of the first conference was published in 1986 by D. Reidel publishing Co and the proceedings of these biannual events will be published every two years from this date.

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**\*\* Annotation for Keplers Six-Cornered Snowflake [7] \*\***

Well, here we are on New Years Day 1610 and Kepler is talking to his friend and benefactor. They have had some private joke about "nothing", which we are not privy to, and Kepler is trying to think of an appropriate gift which will be practically nothing. Firstly he rejects Epicurus' atoms, then small animals such as the mite. Suddenly it begins to snow and a single flake settles on his shoulder. And so begins one of the **most remarkable intellectual meditations ever recorded!**

He knows that a flake comes in several forms and that always it has six corners **but why this sixthness**. This starts him off on considerations of space filling and close packing in nature and we are then shown exactly how this great mind attacks this problem. His meditation, for the first time, gives us the beginnings of a mathematical theory of the genesis of form in nature. Kepler's contribution to the mechanism of hexagonal symmetry in snow was not to be replaced for three hundred years.

The book is a rather wonderful presentation based on the original Latin edition *Strena Seu de Niva Sexangula*, published in Frankfort in 1611. It has been edited very creatively

by L.L. Whyte, a self effacing Oxford scholar (who does not even include his name in the title sheet)! and must be regarded as a model for all such translations of the classics. Firstly there is a synopsis of the Latin text nicely summarising Kepler's important points. Then follows a modernised Latin text with an English translation on the opposite page (by Colin Hardie). Then follow notes on the text. Finally there are two delightful essays quite in the character of Kepler's monograph. The first by J. Mason "On the Shapes of Snow Chrystals" discusses the scientific meaning and validity of Kepler's arguments, and their relation to the history of chrystallography and of space filling. The second essay is by the editor and amongst other things is concerned with Kepler's *facultas formatrix* - **the formative process** - which in modern terms is a comprehensive theory of complex partially ordered systems showing how and why they move towards equilibrium states. Whyte also shows how this process is related to the history of philosophical and scientific ideas on the genesis of forms and gives a summary of the problem of "sixthness" from 1611 to 1962.