On the form of an odd perfect number

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It has been known since the time of Euler that an odd perfect number N (if it exists) must have the form $N = p^a Q^2$ where p is prime and $p = a = 1 \bmod 4$ (see, e.g., [1, pp. 3–33]). Further, it has been shown that N must equal $1 \bmod 12$, or $9 \bmod 36$ [3], [2]. However, we can do a little better than this.

From either result it is immediately evident that if 3 divides N, then 3^k divides N, where $k = 0 \bmod 2$.

If $k = 0$, then N must be of the form $1 \bmod 12$.

For any positive integer $N = p_1^{k1} p_2^{k2} \dots p_n^{kn}$, the sum S of all of the divisors (including 1 and N itself), is given by

$$S = (1 + p_1^1 + p_1^2 + \dots p_1^{k1})(1 + p_2^1 + p_2^2 + \dots p_2^{k2}) \dots (1 + p_n^1 + p_n^2 + \dots p_n^{kn})\,.$$

If N is perfect, it is equal to the sum of its divisors (excepting itself), so $N = S - N$, so $2N = S$. Thus, if N is perfect, and a factor of N is 3^k, then N is itself divisible by $(1 + 3^1 + 3^2 + \dots + 3^k)$.

If $k = 2$, then N must be of the form $9 \bmod 36$. Further, since N is perfect, from the above we know that $3^0 + 3^1 + 3^2 = 1 + 3 + 9 = 13$ must divide $2N$, and hence $N = 0 \bmod 13$. Thus, N must satisfy both $N = 9 \bmod 36$ and $N = 0 \bmod 13$. From the Chinese remainder theorem, we can deduce that N must equal $117 \bmod 468$.

If $k > 2$, then N is divisible by $3^4 = 81$. Thus, N must satisfy both $N = 9 \bmod 36$ and $N = 0 \bmod 81$. From the Chinese remainder theorem, we can deduce that N must equal $81 \bmod 324$.

Thus, if N is an odd perfect number, it must be of the form $N = 1 \bmod 12$ or $N = 117 \bmod 468$ or $N = 81 \bmod 324$.

Of course, it is possible to further refine the last of these results in a similar way, by considering separately values of k greater than or equal to 4.

References

[1] Dickson, L.E. (2005). *History of the Theory of Numbers*, Vol. 1, Divisibility and Primality. Dover, New York.

[2] Holdener, J.A. (2002). A theorem of Touchard and the form of odd perfect numbers. *American Mathematical Monthly* **109,** 661–663.

[3] Touchard, J. (1953). On prime numbers and perfect numbers. *Scripta Mathematica* **19,** 35–39.

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