Book reviews

Spectral Properties of Noncommuting Operators Lecture Notes in Mathematics 1843

Brian Jefferies
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There are many ways to form a function f of a matrix L, or more generally of an operator L acting on a Banach space $\mathcal{X}$. If f is the polynomial $f(z) = \sum_{k=0}^{N} c_k z^k$, then just define $f(L) = \sum_{k=0}^{N} c_k L^k$. Slightly more generally, suppose a holomorphic function f is defined by $f(z) = \sum_{k=0}^{\infty} c_k z^k$ when $|z| < R$, where the radius of convergence R is larger than the norm $||L||$ of L. Then set $f(L) = \sum_{k=0}^{\infty} c_k L^k$. Much more general is the Riesz–Dunford functional calculus. Suppose f is a holomorphic function defined on a neighbourhood Ω of the spectrum $\sigma(L)$ of L. The spectrum is the compact set of all λ in the complex plane $\mathbb{C}$ for which the resolvent $R_L(\lambda) = (\lambda I - L)^{-1}$ does not exist as a bounded operator on $\mathcal{X}$. (For a matrix L, this is just the set of eigenvalues of L.) Then define $f(L) = \frac{1}{2\pi i}\int_\gamma R_L(\zeta) f(\zeta)\, d\zeta$ where γ is a curve in Ω which winds anticlockwise around $\sigma(L)$. These definitions are all consistent, they have natural properties such as $(\alpha f + \beta g)(L) = \alpha f(L) + \beta g(L)$, $(fg)(L) = f(L)g(L)$ and $\sigma(f(L)) = f(\sigma(L))$, and moreover are useful in a variety of contexts. A particular mapping $f \mapsto f(L)$ is called a functional calculus of L.

When $\mathbf{L} = (L_1, L_2, \dots L_n)$ is a commuting family of matrices or operators, then it is clear how to define $f(\mathbf{L})$ when f is a polynomial or power series in n variables. However there are various approaches to defining a joint spectrum $\sigma(\mathbf{L})$ of L, and to constructing a functional calculus of $\mathbf{L}$. When the operators satisfy the property that $\sigma(< \xi, \mathbf{L} >) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$, then one possibility is to use Clifford algebras, and to replace the use of the Cauchy integral in the Riesz–Dunford functional calculus by a higher dimensional Clifford–Cauchy integral.

When the operators L_j do not commute with one another, then it becomes more difficult to construct a reasonable theory. Nevertheless, this is important. For example, the Weyl functional calculus was introduced to consider functions of position (Q_j) and momentum (P_j) under the canonical commutation relations $Q_jP_j - P_jQ_j = i\hbar I$ of quantum theory. In the Weyl functional calculus, one takes symmetric products in forming polynomials. For example, if $\mathbf{L} = (L_1, L_2)$ and $f(x_1, x_2) = x_1x_2$, then $f(\mathbf{L}) = \frac{1}{2}(L_1L_2 + L_2L_1)$. The Clifford approach generalizes quite well to this context, though not easily, and gives us the opportunity to extend the Weyl functional calculus to more general situations, and to identify a type of spectral set as the support of the functional calculus.

A seminal paper on the use of Clifford analysis to study functional calculi of commuting operators is: Alan McIntosh and Alan Pryde, *A functional calculus for several commuting operators*, Indiana University Math. Journal **36** (1987), 421–439; while a seminal paper on non-commuting operators is: Brian Jefferies, Alan McIntosh and James Picton–Warlow, *The monogenic functional calculus*, Studia Mathematica **136** (1999), 99–119.

Rather than say more, let me urge you to read the very clear account by Brian Jefferies in the book under review. Jefferies tells us about the background material, and presents all definitions and results in a precise way which is a pleasure to read. He does not give detailed proofs of all those theorems which are already in the published literature, though, on a number of occasions, he does prove results which go beyond what was previously known, sometimes in important ways.

These new results apply even in the simplest noncommuting situation, namely when $\mathbf{L} = (L_1, L_2, \dots L_n)$ is a family of hermitian matrices, acting as operators on $\mathbb{C}^n$, and f is a continuous function from $\mathbb{R}^n$ to $\mathbb{C}$ with Fourier transform $\hat{f} \in L^1(\mathbb{R}^n)$. The Weyl functional calculus gives a natural definition of $f(\mathbf{L})$ as

$$f(\mathbf{L}) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(i < \xi, \mathbf{L} > \hat{f}(\xi)\, d\xi \ .$$

The support $\gamma(\mathbf{L})$ of this functional calculus is a type of spectral set of $\mathbf{L}$. But what is it? The definition using Fourier transforms can give no information other than an estimate of $\sup |\gamma(\mathbf{L})|$ obtained from the Payley–Wiener theorem. Jefferies goes to great pains to show how further information can indeed be obtained by using Clifford analysis. This is very interesting material.

The book extends but is in no way the last word on the subject. In the final chapter there is an explanation of connections between the Weyl calculus and Feynman's operational calculus, and mention of intriguing possibilities for future research. This could be a fruitful field of endeavour.

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Arnold's Problems

V.I. Arnold (Ed.)
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The statement "*To ask the right question is harder than to answer it*" is attributed to Georg Cantor and one might add that to ask such a question is perhaps the most fruitful and creative action leading to new mathematical knowledge. The book edited by Arnold is full of such questions that were posed in the famous Arnold seminars at the Moscow State University spanning a 50 year period ranging from 1956 in the era of Soviet Union through to present day Russia of 2003.

There are without doubt many legends surrounding the discussions of problems raised in Arnold's seminars and subsequent attempts at solving them, and many readers may have their own anecdotes to relate about this.

The wide scope of mathematics that took centre stage at the seminars is amazing — dynamical systems, number theory, group theory, representation theory and combinatorics, classical and functional analysis to name a few. The diversity that was embraced within the seminars attests to a strong belief in the fundamental unity of a mathematical description of the natural world.

It seems that mathematical life in the Soviet Union in the seminars of Arnold and others — Gel'fand, Sinai, Kirillov, Manin, Novikov to name a few — was and perhaps is rather different from that experienced elsewhere and a formative experience for generations of mathematicians. However through this book the outsider from another era may participate, even if in a vicarious way, in the circle of discussions that took place then.

The book is divided into two parts — the first containing the problems posed in chronological order and the second part with the comments on the problems. In this latter part solutions are given where these have been found along with an extensive historical bibliography of work on the particular problem. Otherwise one can gain an immediate and succinct overview of the current status of a particular open problem from the comments part. One novel and fascinating aspect of the book is that while Arnold has edited the work many of the comments are written by researchers who have contributed directly to the understanding of or solutions to the problems. Besides Arnold there are some 58 other contributors, mostly Arnold's former students, but there are others outside of the Moscow school.

Now we wish to get down to discussing some specific problems. Some of the problems are really the enunciation of programs as in problem *1997-9* about the analogies between various mathematical trinities. Others are very concrete and this reviewer liked the inclusion of problems *1984-7* and *1987-12* regarding the study of decompositions of the space of linear complex ordinary differential equations with singularities into isomonodromy classes and the consequent questions — limits of isomonodromic systems with coalescing singular points, namely their versal deformations, bifurcation diagrams, etc. Another beautiful topic, treated in *1991-11*, *1993-33* and *2000-12*, is the generalisation of Gauss-Kuz'min-Khinchin statistics of simple continued fractions of real numbers in the interval $(0, 1)$ to higher dimensional analogues. While the discussion of problems is even-handed it can be patchy or incomplete — for example the *1990-9* problem asking for a precise meaning to M.V. Berry's assertion that the asymptotics of an oscillatory integral, after subtracting all terms polynomial in the wavelength, exhibits exponentially small 'jumps' with an universal form of the error function has no comment. Also the *1998-15* problem of the quaternionic analog of the determinant fails to mention the application of such determinants by F.J. Dyson in 1972 to random matrix theory based on the 1922 work of E.H. Moore.

Every working mathematician will find something of direct value to their own interests and find it an invaluable resource to dip into from time to time. One hopes that this is an on-going project and that updates will make their appearance regularly. Finally it is necessary to end with a caveat to the quotation made at the opening of this review and which serves as a warning —

"*You are never sure whether or not a problem is good unless you actually solve it*" (M. Gromov).

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Algebraic Integrability, Painlevé Geometry and Lie algebras

Mark Adler, Pierre Van Moerbeke and Pol Vanhaecke
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The aim of this book is to explain "how algebraic geometry, Lie theory and Painlevé analysis can be used to explicitly solve integrable differential equations". It covers many important subjects in the theory of integrable systems and can be also used as an introductory teaching textbook for many classical topics like Lie algebras, Poisson manifolds, algebraic and differential geometry. Usually a study of the theory of integrable systems assumes a solid mathematical background and requires serious preliminary studies in many areas of mathematics. One of the main advantage of this book is that the authors almost succeeded to present a material in a self-contained manner with numerous examples. As a result it can be also used as a reference book for many subjects in mathematics.

The book opens with a short introduction to the differential geometry on complex manifolds and Lie groups and algebras. This chapter also contains a Cartan classification of simple Lie algebras and introduction to twisted affine Lie algebras.

The purpose of the next two chapters is to give a comprehensive introduction into the theory of Liouville integrable systems on Poisson manifolds. The authors start with a notation of Poisson manifolds, hamiltonian dynamics on such manifolds and explain how to introduce a natural Poisson structure on the dual of finite-dimensional and infinite-dimensional Lie algebras. Further they explain a concept of Liouville integrability and a notation of Lax operator which is the main ingredient of the modern approach to algebraic integrability. This part of the book completes with the introduction of the r-matrix structure associated with the L-operator algebras.

The second part of the book is devoted to a description of the algebraic completely integrable systems on affine varieties. It starts with the chapter which explains the geometry of Abelian varieties. However, due to a complexity of the subject the material in this chapter assumes some basic knowledges in differential and Riemann geometry. This part of the book can serve as a good source to refresh reader's knowledges in algebraic geometry. Based on this authors introduce algebraic completely integrable systems on affine varieties. This is the central part of the book. Starting with simple examples authors explain necessary conditions for algebraic complete integrability and explain connections with Lax equations with a parameter. They also formulate a complex version of the Liouville theorem.

The next chapter describes a special class of weight homogeneous algebraic completely integrable systems. Such systems "live" in a flat complex n-dimensional space and under the proper choice of coordinates one can reveal the whole geometry of the system and explicitly compute all their characteristics. This part of the book is quite difficult for reading since it combines some facts from algebraic geometry together with computational algorithms more appropriate for physicists. Although authors tried to illustrate the material with some simple examples, you still need to be either a mathematician interested in practical computations or a physicist with a very good mathematical background.

The concluding part of the book contains applications to some particular algebraic completely integrable systems. The first type of examples is related to investigations of integrable geodesic flows on orthogonal group in 4 dimensions. This chapter is of more interest to specialists in this particular area. Then the authors apply the developed techniques to periodic Toda lattice associated with different root systems and investigate their geometry. Surely these examples will attract many scientist's attention working in the area of integrable systems. The last set of examples concerns the theory of integrable tops. Among others authors consider the famous Kowalevski top and analyze its algebraic structure.

In summary, *Algebraic Integrability, Painlevé Geometry and Lie algebras* is a very good book which covers many interesting subjects in modern mathematical physics. Although its complete study will require a substantial amount of time and efforts for a non-specialist, the reader will profit in a much better understanding of the theory of integrable systems, their geometry and applications in both mathematics and physics.

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Tolerance Graphs

M.C. Golumbic and A.N. Trenk
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A graph is called an interval graph if it admits an interval representation, that is, each vertex can be assigned a closed interval on a real line such that two vertices are adjacent if and only if the corresponding intervals have a non-empty intersection. Interval graphs are well known for their applications to scheduling, resource allocation, microbiology, VLSI circuit design, etc., and they form an important class of perfect graphs. As a generalization of interval graphs, tolerance graphs were introduced by the first-named author of this book and Monma in 1982 so as to reflect the flexibility for sharing resources in some application problems. In a tolerance graph, each vertex v can be represented by an interval I_v together with a tolerance t_v, which is a positive real number or ∞, such that two vertices x, y are adjacent if and only if $|I_x \cap I_y| \geq \min\{t_x, t_y\}$. In other words, two vertices are adjacent if and only if the intersection of the corresponding intervals is large enough to "bother" at least one of them.

This monograph is a comprehensive and thorough treatment of tolerance graphs and their generalizations, and is the first book on the topic. It contains major results on tolerance graphs, often with proofs, that were produced during the last more than two decades. Although the book was intended primarily for researchers and graduate students, as stated in the preface, most parts of it are accessible by those who have only limited background in graph theory. A large part of the book is devoted to relationships among various classes of tolerance or generalized tolerance graphs, and their relationships with some well known classes of graphs such as interval graphs, chordal graphs, weakly chordal graphs, permutation graphs, comparability graphs, cocomparability graphs, parallelogram graphs, trapezoid graphs, threshold graphs, etc. (A typical result of this kind may assert that any graph in class A is in class B, or a graph is in class A if and only if it is in the intersection of classes B and C. For example, tolerance graphs are perfect graphs, interval graphs are precisely tolerance graphs with constant tolerances, etc.) Tolerance-related graphs discussed in the book include bounded tolerance graphs, unit tolerance graphs, proper tolerance graphs, interval probe graphs, bitolerance graphs, unit bounded bitolerance graphs, proper bounded tolerance graphs, directed tolerance graphs, and so on. Partially ordered sets, recognition problems and algorithms relating to tolerance graphs are covered in the book as well.

The book starts with an introductory chapter, where tolerance graphs and other well known intersection graphs are introduced. In the next chapter the authors present early work on tolerance graphs, highlighted with a complete hierarchy of classes of perfect graphs. Chapter 3 gives characterizations of trees, cotrees and bipartite graphs which are tolerance graphs. Chapter 4 discusses interval probe graphs, which were introduced in studying physical mappings of DNA, and proves the NP-completeness of the sandwich problem for such graphs. This chapter also gives a hierarchy of interval graphs, interval probe graphs and other families of tolerance graphs. The next two chapters deal with bounded bitolerance orders and unit tolerance orders, respectively. Chapter 7 investigates tolerance-related orders which are comparability invariant. Chapter 8 is devoted to the problem of recognizing bounded bitolerance orders, and Chapter 9 to algorithms on tolerance graphs. In Chapter 10 the authors consider various classes of bounded bitolerance orders and present a hierarchy of them. In the next two chapters they explore generalizations of tolerance graphs in two different directions: in Chapter 11 the real line is replaced by a tree and intervals are replaced by subtrees, and in Chapter 12 the function $\min\{t_x, t_y\}$ is replaced by a symmetric binary function such as $\max\{t_x, t_y\}$, t_x+t_y, symmetric polynomials in t_x and t_y, etc. Chapter 13 deals with directed tolerance graphs, and the last chapter (Chapter 14) lists a number of open problems and further research directions in the area of tolerance graphs.

The book is well written and easy to follow. Definitions, results, algorithms and proofs are explained and presented clearly. A large number of graphs, tables and figures are included, and they are very helpful to understanding the context. Exercises are provided at the end of each chapter, except the final one where 21 research problems are listed. Since the book collects major results on tolerance graphs, it will be indispensable for researchers and graduate students working on tolerance graphs and other related graph classes. It will also be a very useful reference book for anyone who is interested in algorithmic aspects of graph theory. In this regard the reader may use this book in conjunction with the first author's well known book "Algorithmic Graph Theory and Perfect Graphs" (Academic Press, 1980), whose second edition appeared in 2004. The book under review can be tailored to suit a graduate course with emphasis on graph classes and intersection graphs.

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