

On the number of regions in an m -dimensional space cut by n hyperplanes

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Abstract

In this note we provide a uniform approach for the number of bounded regions cut by n hyperplanes in general position in \mathbb{R}^m and the number of regions cut by n great circles in general position in \mathbb{S}^m .

Key words: Hyperplanes in \mathbb{R}^m , n -Cluster, in general position, binomial coefficients.

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1 Introduction

In this note we will use the term *hyperplane* in \mathbb{R}^m to mean an affine subspace of codimension one in \mathbb{R}^m , i.e., an $(m-1)$ -dimensional plane in \mathbb{R}^m which does not necessarily pass through the origin. It is well known that n hyperplanes, in general position in the m -dimensional space \mathbb{R}^m , will divide \mathbb{R}^m into

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m}$$

regions, where $\binom{p}{q} = 0$ if $q > p$. In this note, we will investigate two related problems:

- (1) What is the nature of these regions?
- (2) How many regions occur if these hyperplanes, in general position, happen to contain a common point?

For the first problem, we will show that among the regions cut by n hyperplanes in general position in \mathbb{R}^m , exactly $\binom{n-1}{m}$ are bounded, and we will describe the geometrical nature of the unbounded regions. For the second problem we may assume without loss of generality that the common point of these n hyperplanes is the origin of \mathbb{R}^m , and the problem is equivalent to that of finding the number of regions on an $(m-1)$ -dimensional sphere cut by n great circles. We will prove that there are

$$2 \left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{m-1} \right]$$

such regions. We would like to point out that our main results could be derived from the classical paper of J. Steiner [4]. However, in our paper, we will give a more direct proof, and our method brings out the geometrical relationships among the following three items: (i) the regions cut by n hyperplanes in \mathbb{R}^m , all of which contain a particular point; (ii) the bounded regions cut by n hyperplanes in general position in \mathbb{R}^m ; and (iii) all the regions cut

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by $n - 1$ hyperplanes in general position in \mathbb{R}^{m-1} (see Theorem 1 and Step A in the proof of Theorem 2 below).

2 Preliminaries

Let us first fix our notations. A subset A of \mathbb{R}^m will be called a k -dimensional affine subset of \mathbb{R}^m if for some point $a \in A$, the set $A - a (= \{x - a \mid x \in A\})$ is a k -dimensional subspace of the linear space \mathbb{R}^m . Thus, a k -dimensional affine subset of \mathbb{R}^m is a translate of a k -dimensional subspace of \mathbb{R}^m . In particular, a hyperplane in \mathbb{R}^m is an affine set of codimension one, i.e., an $m - 1$ -dimensional plane in \mathbb{R}^m which does not necessarily pass through the origin. A set \mathcal{S} of hyperplanes in \mathbb{R}^m is said to be in *general position* if for each k , $0 \leq k \leq m$, no $k + 1$ members of \mathcal{S} contain a common $(m - k)$ -dimensional affine subset of \mathbb{R}^m . If a set \mathcal{S} of n hyperplanes in \mathbb{R}^m in general position all contain a common point, we will call such a set an n -cluster. Finally, we will let $G_m(n)$ be the number of regions in \mathbb{R}^m divided by n hyperplanes in general position and $C_m(n)$ be the number of regions divided by an n -cluster in \mathbb{R}^m . If S^{m-1} is an $m - 1$ dimensional sphere centered at the common point of an n -cluster, each member of the cluster will cut S^{m-1} in a great circle. Thus, the number $C_m(n)$ will equal to the number of regions on an $(m - 1)$ sphere S^{m-1} cut by n great circles on S^{m-1} . The following result is well-known (see for instance, [4] and [5], also cf. [1],[2], and [3]).

Proposition 1 $G_m(n) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{m}$.

3 Main results

Our main results may now be summarized in the following two theorems.

Theorem 1 *The number of regions $C_m(n)$, cut by an n -cluster in \mathbb{R}^m (or cut by n great circles in S^{m-1}), is exactly twice $G_{m-1}(n - 1)$, that is*

$$C_m(n) = 2 \left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \dots + \binom{n-1}{m-1} \right].$$

In the following, we will call a region D in \mathbb{R}^m *cone-like* if whenever an $(m - 1)$ -dimensional plane \mathbb{H} cuts D into two components, D_1 and D_2 , such that the cross-section $\mathbb{H} \cap D$ is bounded, one of the two components D_1 and D_2 will be bounded and the other unbounded. For instance, an infinite tube, open at both ends, would not be cone-like.

Theorem 2 *For any set of n planes in general position in \mathbb{R}^m , among the $G_m(n)$ regions cut by these n planes, exactly $\binom{n-1}{m}$ of them are bounded, and each of the unbounded regions in $G_m(n)$ is always cone-like.*

Since the number of bounded regions in an m -dimensional space \mathbb{R}^m is $\binom{n-1}{m}$, for this number to be positive, $n - 1 \geq m$. Hence, there has to be at least $m + 1$ hyperplanes in \mathbb{R}^m before there can be any bounded region at all. Intuitively, therefore, there seems to more unbounded regions than bounded ones when the space \mathbb{R}^m is cut by n hyperplanes. However, we will prove that as the number n of hyperplanes increases, the number of bounded regions increases much faster than that of the unbounded regions. Specifically, we have the following.

Corollary 1 *Consider a set of n planes in general position in an m -dimensional space \mathbb{R}^m . As the number n of the hyperplanes increases, the ratio of the number of bounded regions over that of the unbounded regions approaches to infinity.*

4 Proofs of the main results

Proof of Theorem 1

Since the number of regions $C_m(n)$, cut by an n -cluster in \mathbb{R}^m , is the same as that cut by n great circles in S^{m-1} , we will consider n great circles in general positions in S^{m-1} . Without loss of generality, we may assume that one of these great circles is the equator of S^{m-1} . Then the radial projection will be a bijection between the remaining great circles and the hyperplanes in general positions in the two tangent planes to the two poles of the sphere. This bijection will also carry the regions between the great circles to those between the hyperplanes. Since there are two tangent planes, $C_m(n)$, the number of regions cut by n great circles in S^{m-1} , equals $2G_{m-1}(n-1)$, or by Proposition 1,

$$C_m(n) = 2 \left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{m-1} \right].$$

□

Proof of Theorem 2.

Consider an arbitrary set \mathcal{C}_0 of n planes in general position in \mathbb{R}^m . We first prove that among the $G_m(n)$ regions cut by these n planes, exactly $\binom{n-1}{m}$ of them are bounded. Let us embed \mathbb{R}^m as an m -dimensional plane \mathbb{H} in the space \mathbb{R}^{m+1} so that \mathbb{H} is one unit away from the origin \mathbf{o} of \mathbb{R}^{m+1} . For each $(m-1)$ -dimensional plane \mathbb{P} in \mathbb{H} that is a member of \mathcal{C}_0 , we let \mathbb{P}' be the m -dimensional plane in the space \mathbb{R}^{m+1} that contains both \mathbf{o} and \mathbb{P} . The collection \mathcal{C} of all such \mathbb{P}' 's is an n -cluster in \mathbb{R}^{m+1} since they all contain \mathbf{o} . Furthermore, they cut \mathbb{H} in members of \mathcal{C}_0 . We will now complete this part of our proof in two steps, A and B.

Step A: Among the $G_m(n)$ regions on \mathbb{H} cut by the n planes in \mathcal{C}_0 exactly $G_m(n) - C_m(n)$ of them are bounded.

We can see this as follows. Let \mathbf{v} be the unit normal vector that extends from \mathbf{o} to the plane \mathbb{H} . Now, let \mathbb{H}_0 be the plane $\mathbb{H} - \mathbf{v}$, and for each t , $0 \leq t \leq 1$, let \mathbb{H}_t be the plane $\mathbb{H}_0 + t\mathbf{v}$. Thus, each \mathbb{H}_t is a parallel translate of the plane \mathbb{H} , $\mathbb{H}_1 = \mathbb{H}$, and \mathbb{H}_0 contains the origin \mathbf{o} . Now, for each t , the planes \mathbb{P}' 's in the collection \mathcal{C} intersect \mathbb{H}_t in a collection of m -dimensional planes in general position, and thereby cut \mathbb{H}_t into $G_m(n)$ regions. As t changes, the boundaries of these regions change continually with t . Now, when $t \rightarrow 0$, some of these regions may shrink to a point. At $t = 0$, the n planes in the collection \mathcal{C} intersect \mathbb{H}_0 in an n -cluster, which cuts \mathbb{H}_0 into $C_m(n)$ regions. Thus, when t moves from a positive value to zero, exactly $G_m(n) - C_m(n)$ regions are eliminated. Since the regions must change continuously and all the regions cut by an n -cluster are unbounded, only the bounded regions are eliminated. Thus, among the $G_m(n)$ regions on \mathbb{H} cut by the n planes in \mathcal{C}_0 , exactly $G_m(n) - C_m(n)$ of them are bounded.

Step B: $G_m(n) - C_m(n) = \binom{n-1}{m}$.

From Proposition 1 and Theorem 1

$$G_m(n) - C_m(n) = \sum_{i=0}^m \binom{n}{i} - 2 \sum_{i=0}^{m-1} \binom{n-1}{i}$$

$$\begin{aligned}
 &= \binom{n}{0} + \sum_{i=1}^m \binom{n}{i} - 2 \sum_{i=0}^{m-1} \binom{n-1}{i} \\
 &= \binom{n}{0} + \sum_{i=1}^m \left[\binom{n-1}{i-1} + \binom{n-1}{i} \right] - 2 \sum_{i=0}^{m-1} \binom{n-1}{i}, \quad \text{by Pascal's triangle} \\
 &= \binom{n}{0} + \sum_{i=1}^m \binom{n-1}{i-1} + \sum_{i=1}^m \binom{n-1}{i} - 2 \sum_{i=0}^{m-1} \binom{n-1}{i}
 \end{aligned}$$

Note that

$$\sum_{i=1}^m \binom{n-1}{i-1} = \sum_{j=0}^{m-1} \binom{n-1}{j}$$

and

$$\binom{n}{0} + \sum_{i=1}^m \binom{n-1}{i} = \sum_{i=0}^m \binom{n-1}{i} = \sum_{i=0}^{m-1} \binom{n-1}{i} + \binom{n-1}{m}.$$

By substituting these two in the preceding line, we have $G_m(n) - C_m(n) = \binom{n-1}{m}$, and among the $G_m(n)$ regions, exactly $\binom{n-1}{m}$ of them are bounded.

To see that each of the unbounded regions is always cone-like, consider an arbitrary set \mathcal{C} of n planes in general position in \mathbb{R}^m . Let \mathcal{R} be the collection of regions in \mathbb{R}^m cut by these n planes. Suppose D is an unbounded region in \mathcal{R} . Let \mathbb{H} be an $(m-1)$ -dimensional plane that cuts D into two components, D_1 and D_2 , such that the cross-section $\mathbb{H} \cap D$ is bounded. Since D is itself unbounded, to show that D is cone-like, it is sufficient to show that one of D_1 and D_2 is bounded. Tilting \mathbb{H} slightly if necessary, we may obtain an $(m-1)$ -dimensional plane \mathbb{H}' which cuts D into two components, D'_1 and D'_2 , and which is in general position with respect to the planes in \mathcal{C} . By pushing \mathbb{H}' slightly to one side or the other of \mathbb{H} , we can make $D'_1 \supset D_1$ or $D'_2 \supset D_2$. Thus, if we can show that one of D'_1 and D'_2 is bounded, then one of D_1 and D_2 will also be bounded. In the following we may hence assume that the plane \mathbb{H} is in general position with respect to the planes in \mathcal{C} and consider D_1 and D_2 instead of D'_1 and D'_2 .

Now, $\mathcal{C} \cap \mathbb{H} = \{\mathbb{P} \cap \mathbb{H} \mid \mathbb{P} \in \mathcal{C}\}$ is a set of n planes in the $(n-1)$ space \mathbb{H} that cuts \mathbb{H} into $G_{m-1}(n)$ regions, each of which divides a region of \mathcal{R} into two. Thus, with the addition of \mathbb{H} , $G_{m-1}(n)$ new regions have been added. Now, each of the unbounded regions among the $G_{m-1}(n)$ regions on \mathbb{H} can add only a new unbounded region in \mathbb{R}^m , not a bounded one. Thus, the new bounded regions in \mathbb{R}^m can come only from bounded regions of \mathbb{H} . Before the addition of \mathbb{H} , there were already $\binom{n-1}{m}$ bounded regions in \mathbb{R}^m cut by the members of \mathcal{C} , and after \mathbb{H} is added, there are $\binom{n}{m}$ bounded regions. Thus, there are $\binom{n}{m} - \binom{n-1}{m} = \binom{n-1}{m-1}$ new regions being added. But there are only $\binom{n-1}{m-1}$ bounded regions on \mathbb{H} . Each of these bounded regions must therefore create a new bounded region in \mathbb{R}^m . But the cross-section $\mathbb{H} \cap D$ is one such bounded region which cuts an old region D into two components, D_1 and D_2 . Thus, one of D_1 and D_2 must be bounded. \square

Proof of Corollary 1

Consider a set of n planes in general position in an m -dimensional space \mathbb{R}^m . For a fixed m , the number of bounded regions, $\binom{n-1}{m}$, has an order of magnitude $O(n^m)$. On the other

hand, the number of unbounded region, from A in the proof of Theorem 2, is $C_m(n)$, but

$$C_m(n) = 2 \left[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{m-1} \right].$$

It has an order of magnitude of $O\left(\frac{n-1}{m-1}\right) = O(n^{m-1})$. Thus, the order of magnitude of the bounded regions is greater than that of the unbounded regions and the ratio of these two number approaches infinity as n increases. \square

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