

## Trisecting the equilateral triangle with rational trigonometry

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### What is rational trigonometry?

Rational trigonometry is a dramatically simpler way of studying the measurement of triangles. It replaces the quasi-linear concepts of *distance* and *angle* with the quadratic algebraic concepts of *quadrance* and *spread*, and replaces the usual swag of trigonometric laws with simpler polynomial ones, involving no transcendental functions. It is much easier to learn, more powerful, and more accurate too. This new theory was introduced last year in [2], and has seen a fair amount of internet discussion.

Here we give a simple but instructive example, which showcases the basic laws of the subject and demonstrates the power of the method. Suppose that  $\overline{ABC}$  is an equilateral triangle in which we have trisected each of the sides, and joined all these new points to the opposite vertices as in Figure 3. What are all the angles in this diagram?

Although this appears a reasonable question, it turns out that the answers are not particularly interesting, with approximate values such as  $19.106605^\circ$ ,  $81.786789^\circ$  and  $38.213210^\circ$ . This is why you are probably not familiar with the situation. Let's turn to rational trigonometry to find the correct question, so that the answers are interesting after all.

The **quadrance**  $Q(A_1, A_2)$  between two points in the plane is the square of the distance, so that in terms of coordinates

$$Q(A_1, A_2) = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

The **spread**  $s(l_1, l_2)$  between two lines in the plane is the square of the sine of the angle between them, but we use a better definition. Suppose the lines meet at a point  $A$ , and  $B$  is any other point on either of the two lines, with  $C$  the foot of the perpendicular from  $B$  to the other line as in Figure 1. Then

$$s(l_1, l_2) = \frac{Q(B, C)}{Q(A, B)} = \frac{P}{R}.$$

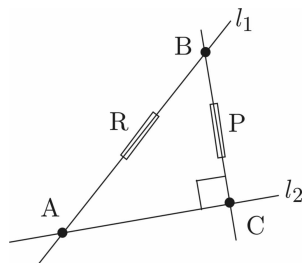


Figure 1. Spread between two lines

If the lines have equations  $a_1x + b_1y + c_1 = 0$  and  $a_2x + b_2y + c_2 = 0$  then the spread is the rational expression

$$s(l_1, l_2) = \frac{(a_1b_2 - a_2b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}.$$

In terms of coordinates of points,

$$s(A_3A_1, A_3A_2) = \frac{((y_1 - y_3)(x_3 - x_2) - (y_2 - y_3)(x_3 - x_1))^2}{((x_1 - x_3)^2 + (y_1 - y_3)^2)((x_2 - x_3)^2 + (y_2 - y_3)^2)}. \quad (1)$$

The spread  $s$  is always a number between 0 and 1, being 0 when the lines are parallel and 1 when the lines are perpendicular. An angle of  $45^\circ$  or  $135^\circ$  is a spread of  $1/2$ , an angle of  $30^\circ$  or  $150^\circ$  is a spread of  $1/4$ , and an angle of  $60^\circ$  or  $120^\circ$  is a spread of  $3/4$ . You can make a spread protractor that measures spreads in the same way that an ordinary protractor measures angles. Here is one you can download from the internet [1].

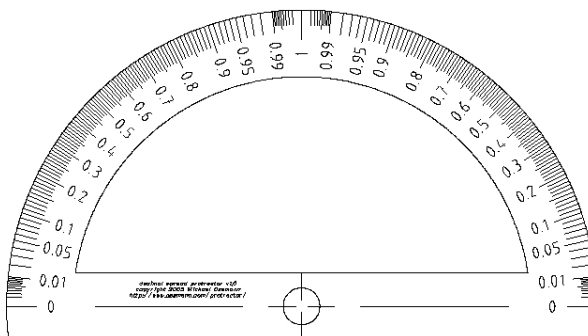


Figure 2. A spread protractor

Of course it takes a bit of getting used to the fact that the spread is not ‘linear’. However the advantages quickly become apparent when we look at the main laws of trigonometry expressed with these concepts.

If a triangle has quadrances  $Q_1, Q_2$  and  $Q_3$ , and corresponding spreads  $s_1, s_2$  and  $s_3$  then the **Spread law** states that

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}$$

while the **Cross law** states that

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3).$$

These are direct analogs of the Sine law and Cosine law. The usual ‘Sum of angles is 180 degrees law’ is replaced by the **Triple spread formula**, which states that

$$(s_1 + s_2 + s_3)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4s_1s_2s_3.$$

The other two main laws are just special cases of the Cross law. When  $s_3 = 0$  the three points are collinear and the three quadrances satisfy  $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2$  or equivalently

$$(Q_1 + Q_2 + Q_3)^2 = 2(Q_1^2 + Q_2^2 + Q_3^2).$$

That’s the **Triple quad formula**. Note that the Triple quad formula and the Triple spread formula differ only by a single cubic term. When  $s_3 = 1$  in the Cross law, you get **Pythagoras’ theorem** in the more basic form  $Q_1 + Q_2 = Q_3$ . A complete derivation of these laws

from first principles takes a few days to explain to an average high school maths student (as opposed to a few months for the current theory of trigonometry).

### Trisecting the equilateral triangle

Label the equilateral triangle as shown, and suppose that the triangle has been scaled so that each of the boundary segments such as  $\overline{AD}$ ,  $\overline{DG}$  and so on have quadrance  $Q = 1$ . This

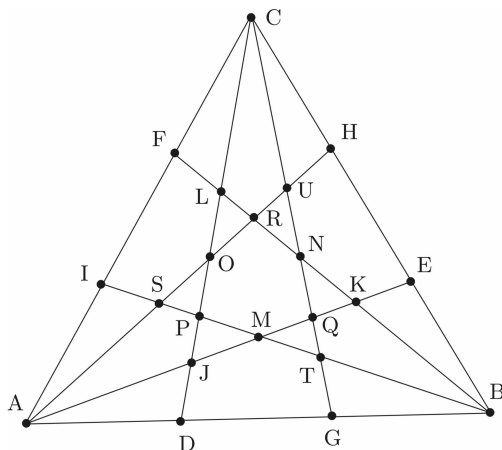


Figure 3. Trisected equilateral triangle

figure also appeared in [3], and has a projective analog that encodes the Platonic solids very efficiently, as I will show elsewhere. We are now going to derive the spreads in the diagram. Symmetry will of course cut down on the possibilities.

The triangle  $\overline{ACD}$  has quadrances  $Q(A, D) = 1$  and  $Q(A, C) = 9$  since quadrance scales quadratically, and spread  $s(AD, AC) = 3/4$ .

The Cross law then asserts that  $Q = Q(C, D)$  satisfies

$$(9 + 1 - Q)^2 = 4 \times 9 \times 1 \times (1 - 3/4)$$

or

$$(Q - 10)^2 = 9.$$

There are two solutions and the relevant one is  $Q = 10 - 3 = 7$  since the other one corresponds to the situation where  $D$  is on the other side of  $A$ . The Spread law asserts that

$$\frac{3/4}{7} = \frac{s(CA, CD)}{1} = \frac{s(DA, DC)}{9}$$

so that  $s(CA, CD) = 3/28$  and  $s(DA, DC) = 27/28$ .

The Spread law applied now to the triangle  $\overline{ACG}$  gives

$$\frac{3/4}{7} = \frac{27/28}{9} = \frac{s(CA, CG)}{4}$$

so that  $s(CA, CG) = 3/7$ .

Both triangles  $\overline{ADJ}$  and  $\overline{AGQ}$  have spreads of  $3/28$  and  $27/28$ , so the third spread  $s$  must satisfy the Triple spread formula

$$\left(\frac{3}{28} + \frac{27}{28} + s\right)^2 = 2 \left( \left(\frac{3}{28}\right)^2 + \left(\frac{27}{28}\right)^2 + s^2 \right) + 4 \times \frac{3}{28} \times \frac{27}{28} \times s.$$

This is a quadratic equation which turns out to factor as

$$(4s - 3)(49s - 48) = 0.$$

Thus  $s(JA, JD) = 3/4$  (as we could have also seen by symmetry since  $\overline{JKL}$  is an equilateral triangle) and  $s(QA, QG) = 48/49$ .

The Triple spread formula simplifies in a pleasant way when two of the spreads  $s$  are equal—in this case the third spread is either 0 or  $4s(1 - s)$ . Thus in the isosceles triangle  $\overline{ABM}$

$$s(MA, MB) = 4 \times \frac{3}{28} \times \left(1 - \frac{3}{28}\right) = \frac{75}{196}.$$

Similarly in the isosceles triangle  $\overline{AEH}$

$$s(AE, AH) = 4 \times \frac{27}{28} \times \left(1 - \frac{27}{28}\right) = \frac{27}{196}.$$

So we get the following picture of spreads.

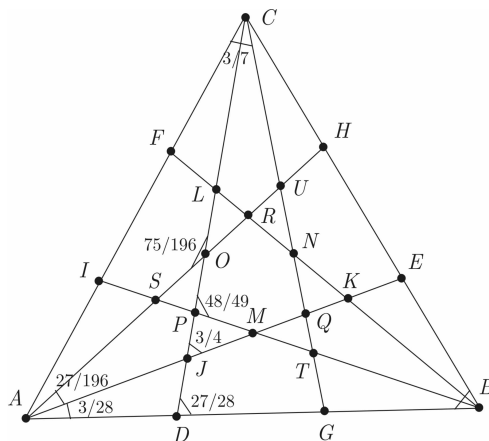


Figure 4. Spreads in the trisected equilateral triangle

It is now possible to find all the quadrances in the diagram too—just use the Spread law repeatedly. Of particular interest is the fact that

$$Q(A, J) = Q(J, K) = Q(B, K) = Q(K, L) = Q(C, L) = Q(L, J).$$

This property characterizes the spherical analog of this situation which is involved in the construction of the Platonic solids.

Classical trigonometry relies heavily on  $90^\circ - 45^\circ - 45^\circ$  and  $90^\circ - 60^\circ - 30^\circ$  triangles for examples and test questions, as these are largely the only ones that students can calculate with directly by hand. Once you make the switch to rational trigonometry, the opportunities widen immensely. For example, the triangle  $\overline{AGU}$  in the previous diagram has spreads  $3/7, 27/28$  and  $3/4$ . With an appropriate scaling, the quadrances may be taken to be 4, 9 and 7. This is now a triangle that you can use for demonstrations and further analysis.

As an aside, there are somewhat interesting relationships between the areas of the various polygons in this figure. You might enjoy proving for example that the area of the interior hexagon  $\overline{OPMQNR}$  is one-tenth the area of the original triangle  $\overline{ABC}$ . This is an affine property, meaning that it is preserved under affine transformations, and so holds rather generally.

## Geometry over other fields

This example has shown that rational trigonometry often involves solving quadratic equations, and that two possible solutions may arise from the same initial data. In [2] there are some additional laws, called the *Triangle spread rules*, that exploit convexity over the ‘real numbers’ to specify exactly which solution to take. They are a bit too involved to state here.

However it should be noted that rational trigonometry allows one to study *metrical geometry* over arbitrary fields, and in general different solutions to a quadratic equation correspond to alternative configurations.

In particular the equilateral trisection problem can also be studied in other fields if equilateral triangles exist, for which it is necessary that 3 be a square. Now in  $\mathbb{F}_{11}$  the number  $3 = 5^2$  is a square, and setting

$$A = [0, 0] \quad B = [3, 0] \quad C = [7, 2]$$

you get

$$Q(A, B) = Q(B, C) = Q(A, C) = 9,$$

so that  $\overline{ABC}$  is equilateral. Furthermore we may trisect the sides, taking

$$\begin{array}{lll} D = [1, 0] & F = [1, 5] & H = [2, 5] \\ E = [8, 8] & G = [2, 0] & I = [6, 8]. \end{array}$$

Then using the formula (1) and working in  $\mathbb{F}_{11}$ , you can check that

$$s(CA, CD) = 6 = \frac{3}{28} \quad s(DA, DC) = 10 = \frac{27}{28} \quad s(CA, CG) = 2 = \frac{3}{7}$$

and so on.

To give a further example, the point  $U$  is  $[8, 9]$ , and the triangle  $\overline{AGU}$  has quadrances 7, 2 and 4, and spreads 2, 10 and 9. Its circumcenter is  $C = [1, 9]$ , its centroid is  $[7, 3]$  and its orthocenter is  $O = [8, 2]$ . These points are collinear, since

$$\frac{2}{3}[1, 9] + \frac{1}{3}[8, 2] = [7, 3].$$

Furthermore the center of the nine point circle of  $\overline{AGU}$  is  $N = [10, 0]$ , which is the midpoint of  $O$  and  $C$ . Thus the usual relations in the Euler line of a triangle are alive and well.

Don't be fooled by the simplicity of the above calculations. Rational trigonometry and its extension to *Universal Geometry* transform many aspects of metrical geometry and introduce new directions for algebraic geometry. The theory extends not only to general fields, but also to arbitrary quadratic forms, with a projective version that redefines both spherical and hyperbolic geometries, as described in [4].

## References

- [1] M. Ossmann, ‘Print a Protractor’, available online at <http://www.ossmann.com/protractor>.
- [2] N.J. Wildberger, *Divine Proportions: Rational Trigonometry to Universal Geometry* (Wild Egg Books Sydney 2005).
- [3] N.J. Wildberger, *Survivor: The Trigonometry Challenge*, posted 2006 at <http://wildegg.com/authors.htm>.
- [4] N.J. Wildberger, *Affine and Projective Universal Geometry*, submitted 2006.

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