Product constructions for transitive decompositions of graphs

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Abstract

A *decomposition* of a graph is a partition of the edge set, giving a set of subgraphs. A *transitive decomposition* is a decomposition which is highly symmetrical, in the sense that the subgraphs are preserved and transitively permuted by a group of automorphisms of the graph. This paper describes some 'product' constructions for transitive decompositions of graphs, and shows how these may be used to characterise transitive decompositions where the group has a particular type of rank 3 action on the graph.

1 Introduction

Any graph can be broken down into subgraphs by partitioning the edge set, giving what is known as a graph *decomposition*. This idea originated in the work of Julius Petersen and others in the early 1900s, and since then has developed into a broad and elaborate branch of graph theory, with applications in areas such as coding theory, experimental design, crystallography and the design of communications networks. In the last few decades there has been increasing interest in the study of graph decompositions from an algebraic point of view. Examples of this approach can be found in the theory of homogeneous factorisations of graphs [8, 3], and also in research devoted to classifying and characterising certain families of graph decompositions (see for example [9], [6] and [7]). All of these examples have an important feature in common: namely, that the decompositions in question are 'highly symmetrical' in a well-defined group-theoretical sense. It is this observation that gave rise to the general notion of a *transitive decomposition*. Essentially, this refers to a graph decomposition whose subgraphs are 'preserved' as a set and permuted transitively by a group of graph automorphisms.

Formally, a decomposition of a graph Γ is expressed as a partition $\mathcal{P}$ of the edge set of Γ, where each part P in the partition corresponds to a subgraph Γ_P whose edge set is P and whose vertex set consists of those vertices of Γ incident with edges in P. A transitive decomposition is defined with respect to a group G of automorphisms of Γ (in other words, a group of permutations of the vertex set of Γ which leave the edge set invariant). In particular, we require that the following two conditions hold (note that in general a permutation group is *transitive* on a set Ω if for all $\omega, \omega' \in \Omega$, there is a permutation in the group mapping ω to ω'):

(i) given any subgraph in the decomposition and any automorphism in G, the subgraph is mapped by the automorphism either wholly to itself or wholly to a different subgraph in the decomposition (in other words, G preserves $\mathcal{P}$); and

(ii) given any pair of subgraphs in the decomposition, there is an automorphism in G mapping the first wholly to the second (in other words, G is transitive on $\mathcal{P}$).

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When these conditions hold, we call the triple $(G, \Gamma, \mathcal{P})$ a transitive decomposition. A more general definition allows $\mathcal{P}$ to be a partition of the arcs (ordered pairs of adjacent vertices) of Γ; however for simplicity we will focus on the case where $\mathcal{P}$ is an edge partition.

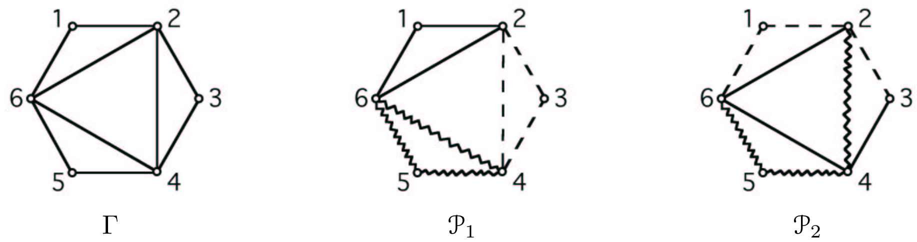

Figure 1. A graph Γ and two different partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of its edge set. Each of the partitions breaks Γ into three subgraphs: one with solid edges, one with dotted edges, and one with zigzag edges.

Let G denote the full automorphism group of the graph Γ in Figure 1. The automorphisms in G can be thought of as 'symmetries' of Γ, and each can be represented by one of the following:

- a flip about an axis through opposite vertices; or
- a rotation clockwise by 0, 2 or 4 vertices.

With this in mind, it is not hard to check that condition (i) above holds for the decomposition $\mathcal{P}_1$ in the middle of Figure 1: for example, the flip about the axis through the vertices 1 and 4 leaves the solid subgraph where it is, and swaps the dotted subgraph wholly with the zigzag subgraph – and any other automorphism in G preserves the set of subgraphs in a similar way. It is also easy to see that for any pair of subgraphs, there is an automorphism mapping the first to the second: if we choose, for example, the solid and the dotted subgraphs, then a rotation by 2 vertices clockwise is such an automorphism. Since conditions (i) and (ii) are satisfied, the triple $(G, \Gamma, \mathcal{P}_1)$ is a transitive decomposition.

On the other hand, $(G, \Gamma, \mathcal{P}_2)$ is *not* a transitive decomposition. This can be checked easily by noting that, in the picture on the far right of Figure 1, no automorphism in G will map (for example) the dotted subgraph to the solid subgraph, and so condition (ii) does not hold.

One of the most interesting reasons for studying transitive decompositions is the large number of connections they have with other structures in combinatorics and geometry. For example, a *partial linear space* consists of a set of *points* together with a set of *lines* (subsets of points) with the property that every pair of points is in at most one line. It turns out that every partial linear space whose automorphism group acts transitively on lines is equivalent to a transitive decomposition of a graph into *complete subgraphs* (in other words, subgraphs in which every pair of vertices is an edge). The points of the partial linear space correspond to the vertices of the graph, and the lines correspond to the vertex sets of the subgraphs. A proof of this correspondence appears in [2], and Figure 2 shows an example.

Transitive decompositions also have an interesting application in the colouring of modular origami models, described in [2].

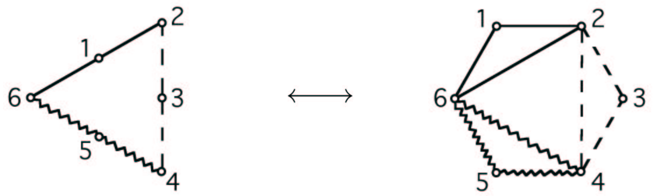

Figure 2. A line transitive partial linear space (on the left), and the corresponding graph (on the right). The partial linear space has 3 lines: a solid line, a dotted line and a zigzag line, and these correspond respectively to the solid, dotted and zigzag subgraphs on the right, giving a transitive decomposition of the graph. Note that each such subgraph in the decomposition is a complete graph with 3 vertices.

2 Product Constructions

In this section we give an overview of some product constructions for transitive decompositions. These constructions are described in more detail in [1], where they are used in an application to rank 3 transitive decompositions (outlined in Section 3 of this paper).

There are several well-known methods of constructing graphs as 'products' of smaller graphs. For example, given a graph Δ (with vertex set $V\Delta$ and edge set $E\Delta$), the *Cartesian product* $\Delta\Box\Delta$ has a 'grid' structure consisting of flattened 'horizontal' copies of Δ overlayed on flattened 'vertical' copies, as shown in Figure 3. The vertex set of $\Delta\Box\Delta$ is $V\Delta \times V\Delta$, and the edges $\{(\alpha_1, \alpha_2), (\beta_1, \beta_2)\}$ are of two types:

- those with $\{\alpha_2, \beta_2\} \in E\Delta$ and $\alpha_1 = \beta_1$ ('horizontal' edges); and
- those with $\{\alpha_1, \beta_1\} \in E\Delta$ and $\alpha_2 = \beta_2$ ('vertical' edges).

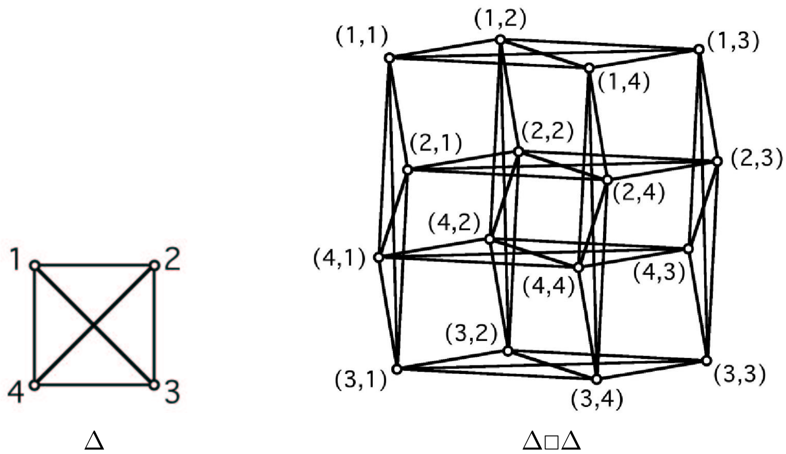

Figure 3. A graph Δ and the Cartesian product $\Delta\Box\Delta$, drawn to show the flattened 'horizontal' and 'vertical' copies of Δ.

At this point we ask an obvious question: can we use a transitive decomposition of Δ to construct a transitive decomposition of $\Delta\Box\Delta$? In fact (assuming certain conditions hold) there are several construction methods which achieve this, and we describe a couple of them below. First note that if H is a group of automorphisms of Δ, then the *wreath product* $H \wr S_2$ (acting in *product action*, where S_2 denotes the symmetric group of degree 2) is a group of automorphisms of $\Delta \square \Delta$. Essentially, this group consists of combinations of three different types of automorphisms of $\Delta \square \Delta$: automorphisms in H acting synchronously on the horizontal copies of Δ, automorphisms in H acting synchronously on the vertical copies, and a 'flip about the diagonal' which interchanges the horizontal copies with the vertical copies.

Now, let $(H, \Delta, \mathcal{Q})$ be a transitive decomposition. We construct a partition of the edge set of $\Delta \square \Delta$ as follows. First, we partition the edges of each of the horizontal copies of Δ according to the partition $\mathcal{Q}$. Second, we partition the edges of the vertical copies according to $\mathcal{Q}$ but using a *different 'colour' scheme* from that used for the horizontal copies. This gives a partition $\mathcal{P}$ in which each part is, for a given $Q \in \mathcal{Q}$, either a union of the copies of Q from each of the horizontal copies of Δ, or a union of the copies of Q from each of the vertical copies of Δ (see Figure 4). Construction 1 gives a formal description.

Construction 1 Let $(H, \Delta, \mathcal{Q})$ be a transitive decomposition. For each $Q \in \mathcal{Q}$ define a subset $\square_1 Q$ of $E(\Delta \square \Delta)$ consisting of all edges $\{(\alpha_1, \alpha_2), (\beta_1, \beta_2)\}$ with $\{\alpha_1, \beta_1\} \in Q$ and $\alpha_2 = \beta_2$; and define also a subset $\square_2 Q$ consisting of all edges $\{(\alpha_1, \alpha_2), (\beta_1, \beta_2)\}$ with $\{\alpha_2, \beta_2\} \in Q$ and $\alpha_1 = \beta_1$. Let $\mathcal{P}$ be the set of all subsets $\square_1 Q$ and $\square_2 Q$ for all $Q \in \mathcal{Q}$.

With $\mathcal{P}$ as in Construction 1, $(H \wr S_2, \Delta \square \Delta, \mathcal{P})$ is a transitive decomposition. A more general version of this result is proved in [1].

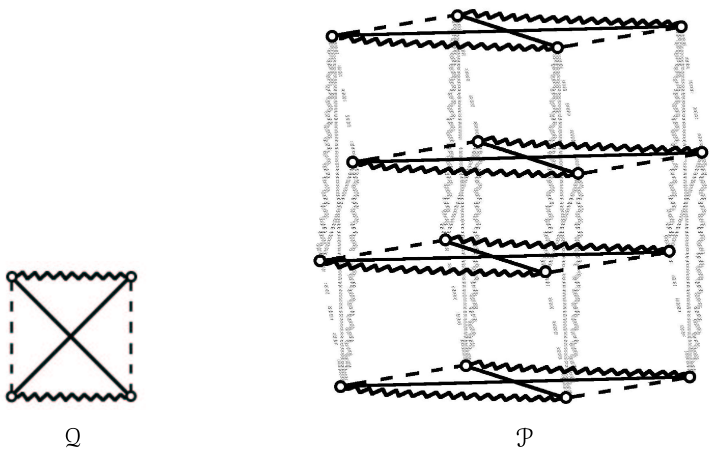

Figure 4. An edge partition $\mathcal{Q}$ of K_4 and the edge partition $\mathcal{P}$ of $K_4 \square K_4$ obtained using Construction 1. Edges coloured grey are in different parts from those coloured black. Note that $(S_4, K_4, \mathcal{Q})$ is a transitive decomposition (where S_4 is the automorphism group of K_4); and so by the comment following Construction 1, $(S_4 \wr S_2, K_4 \square K_4, \mathcal{P})$ is also a transitive decomposition.

Another similar construction involves partitioning the edges of the copies of Δ according to $\mathcal{Q}$, but in this case using a different 'colour' scheme for *each* horizontal and *each* vertical copy of Δ. So each part in the partition of $E(\Delta \square \Delta)$ is, for some $Q \in \mathcal{Q}$, the copy of Q from either a single horizontal or a single vertical copy of Δ. Construction 2 gives a formal description.

Construction 2 Let $(H, \Delta, \mathcal{Q})$ be a transitive decomposition. For each $Q \in \mathcal{Q}$ and each $\delta \in V\Delta$ define a subset $\square_{1,\delta}Q$ of $\Delta\square\Delta$ consisting of all edges $\{(\alpha_1, \alpha_2), (\beta_1, \beta_2)\}$ with $\{\alpha_1, \beta_1\} \in Q$ and $\alpha_2 = \beta_2 = \delta$; and define also a subset $\square_{2,\delta}Q$ consisting of all edges $\{(\alpha_1, \alpha_2), (\beta_1, \beta_2)\}$ with $\{\alpha_2, \beta_2\} \in Q$ and $\alpha_1 = \beta_1 = \delta$. Let $\mathcal{P}$ be the set of all subsets $\square_{1,\delta}Q$ and $\square_{2,\delta}Q$ for all $Q \in \mathcal{Q}$ and all $\delta \in V\Delta$.

With $\mathcal{P}$ as in Construction 2, $(H \wr S_2, \Delta\square\Delta, \mathcal{P})$ is a transitive decomposition *if and only if H acts transitively on the vertices of* Δ. Again, a more general version of this result is proved in [1].

3 An application to rank 3 transitive decompositions

A permutation group G acting on a set Ω is said to have *rank 3* if G has exactly 3 orbits in its induced action on $\Omega \times \Omega$ (that is, in the action given by $(\alpha, \beta)^g = (\alpha^g, \beta^g)$ for all $(\alpha, \beta) \in \Omega \times \Omega$ and $g \in G$). This idea can be understood very naturally in the context of graph theory: given a graph Γ (which is non-complete and which has a non-empty arc set), a group G of automorphisms of Γ has rank 3 on the vertices of Γ if and only if

- (i) G is transitive on the vertices of Γ;
- (ii) G is transitive on the arcs of Γ (ordered pairs of adjacent vertices); and
- (iii) G is transitive on the 'non-arcs' of Γ (ordered pairs of non-adjacent vertices).

Permutation groups with a *primitive* rank 3 action are well-understood as a result of classifications carried out by Liebeck and Saxl in the 1980s (see [10] and [11]), and this has stimulated a number of authors to study and classify mathematical structures preserved by a rank 3 group. Some of these structures have close connections with transitive decompositions (such as line transitive partial linear spaces, as explained earlier). Consequently, the study of transitive decompositions with a rank 3 group can often yield useful information about these other rank 3 structures.

Primitive rank 3 groups come in three different types: *almost simple* type, *affine* type and *product action* type. It is mainly when studying transitive decompositions with a group in the last case that the product constructions in Section 2 are useful. A primitive rank 3 group of product action type can act as a group of automorphisms of the Cartesian product $K_m\square K_m$ (where K_m is the complete graph with m vertices) or of the complement of $K_m\square K_m$ (which may be expressed as a different kind of graph product – see [1]). The problem of characterising the associated transitive decompositions is then as follows: given a transitive decomposition $(G, \Gamma, \mathcal{P})$ where G is a primitive rank 3 group of product action type, and where Γ is either $K_m\square K_m$ or its complement, what are the possibilities for $\mathcal{P}$? As it turns out, $\mathcal{P}$ can be characterised in this case using product constructions similar to the ones given in Section 2. This is explained in detail in [1], leading to the following theorem.

Theorem 1 *Suppose that* $(G, \Gamma, \mathcal{P})$ *is a transitive decomposition where* G *is a primitive rank 3 group of product action type. Then there exists a transitive decomposition* $(H, K_m, \mathcal{Q})$*, where* H *is transitive on the arcs of* K_m*, such that* $\mathcal{P}$ *arises from* $\mathcal{Q}$ *by one of five explicit 'product' constructions.*

This result states that all methods for *constructing* the partitions for the transitive decompositions in the statement are known. The input for each of the constructions in this case is a particular type of transitive decomposition $(H, K_m, \mathcal{Q})$ where H acts transitively on the arcs of K_m. These transitive decompositions have been classified by Thomas Sibley in [7]. Consequently, all transitive decompositions $(G, \Gamma, \mathcal{P})$ where G is a primitive rank 3 group of product action type are well understood.

Theorem 1 has some interesting implications for the combinatorial and geometrical structures related to these rank 3 transitive decompositions. For example, the following corollary arises as a consequence of the connection between transitive decompositions and line transitive partial linear spaces outlined at the end of Section 1. An equivalent result is proved using different methods by Devillers in [4].

Corollary 1 *Suppose that a partial linear space $\mathcal{L}$ is preserved by a primitive rank 3 group of product action type. Then $\mathcal{L}$ may be constructed from a 2-transitive linear space (all examples of which are classified in [5]).*

There is a family of primitive rank 3 groups of affine type which can also act as automorphism groups of $K_m \square K_m$ and its complement. At present it appears that most transitive decompositions with such a rank 3 group can be characterised in a similar way to those where the group is of product action type; however the constructions and proofs involved are more complicated.

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