Late last year several news-items appeared announcing the solution of the 30 year old Lindner's conjecture by UoQ Graduate student Daniel Horsley and his supervisor Darryn Bryant, see e.g.,

http://www.uq.edu.au/news/?article=8359,
http://www.epsa.uq.edu.au/?page=40325&pid=7452,
http://www.questnews.com.au/article/2005/11/30/5392_wn_news.html.

The next article by Darryn and Daniel provides an introduction to the Lindner conjecture and its solution.

The embedding problem for partial Steiner triple systems

Darryn Bryant and Daniel Horsley

1 Introduction

This article describes Lindner's embedding problem for partial Steiner triple systems [13], surveys some of the history of the problem, and briefly discusses its recent solution [3]. Consider the following system of seven 3-element subsets of $V = \{1, 2, 3, 4, 5, 6, 7\}$.

$$\{1,2,3\} \quad \{1,4,5\} \quad \{1,6,7\} \quad \{2,4,6\} \quad \{2,5,7\} \quad \{3,4,7\} \quad \{3,5,6\}$$

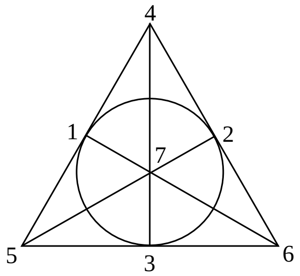
The system has the nice property that any pair of distinct elements of V occurs in exactly one of the subsets. This makes it an example of a *Steiner triple system*. Steiner triple systems first appeared in the mathematical literature in the mid-nineteenth century but the concept must surely have been thought of long before then. An excellent historical introduction appears in [7]. As pointed out there, it is interesting that "triple systems find their origins in studies of cubic curves, rather than in recreational problems as is often thought".

A Steiner triple system is more formally defined as a pair $(V, \mathcal{B})$ where V is a finite set and $\mathcal{B}$ is a set of 3-element subsets of V such that each 2-element subset of V is a subset of exactly one of the 3-element subsets in $\mathcal{B}$. The elements of $\mathcal{B}$ are called *triples* and $|V|$ is the *order* of the system. If there is a Steiner triple system of order v then simple counting establishes that it contains $v(v-1)/6$ triples, and each element of v occurs in $(v-1)/2$ triples. It follows that if there is a Steiner triple system of order v, then $v \equiv 1 \text{ or } 3 \pmod 6$. Such integers are called *admissible*. In 1847 Kirkman [11] proved the existence of Steiner triple systems of all admissible orders. Steiner triple systems of orders 1 and 3 are trivial.

Up to isomorphism, the number $N(v)$ of Steiner triple systems of order v for $v = 7, 9, 13, 15, 19$ is given in the following table, see [7, 10]. For $v > 19$ the exact value of $N(v)$ is unknown.

v	7	9	13	15	19
$N(v)$	1	1	2	80	11,084,874,829

The Steiner triple of order 7 was given above and is equivalent to the Fano plane or projective plane of order 2 (see the figure below left). The Steiner triple system of order 9 is equivalent to the affine plane of order 3 and can be presented as the rows, columns, diagonals and reverse diagonals of the matrix M shown below right.



$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{array}{cccc} \{1,2,3\} & \{1,4,7\} & \{1,5,9\} & \{1,6,8\} \\ \{4,5,6\} & \{2,5,8\} & \{2,6,7\} & \{2,4,9\} \\ \{7,8,9\} & \{3,6,9\} & \{3,4,8\} & \{3,5,7\} \end{array}$$

Steiner triple systems and their generalisations have been the focus of over 1000 research articles and there are now many different methods for constructing them. They are used in cryptography, coding theory, group testing, experimental design et cetera, see [6, 14] for example.

2 Partial Systems and Embedding

Given the well-established existence results for Steiner triple systems, it is natural to ask about systems where some triples are specified in advance. That is, to ask for an *embedding* of the specified triples in a Steiner triple system. For example, can the three triples $\{1,2,3\}, \{1,4,5\}$ and $\{2,4,6\}$ be embedded in a Steiner triple system of order 7? What about the two triples $\{1,2,3\}$ and $\{4,5,6\}$? The answer to the first question is "yes". The three triples are embedded in the Steiner triple system of order 7 given at the beginning of the article. The answer to the second question is "no". It is a simple exercise to check this directly, but it also follows from the fact that (up to isomorphism) there is only one Steiner triple system of order 7, and it has the property that every pair of distinct triples intersects in exactly one element. However, $\{1,2,3\}$ and $\{4,5,6\}$ can be embedded in a Steiner triple system of order 9, for example the one given above.

Clearly if two distinct triples intersecting in more than one element are specified, then no embedding is possible. A *partial Steiner triple system* is thus defined as a set of triples with the property that any two distinct triples intersect in at most one element. The number of distinct elements occurring in the triples is the *order* of the partial system.

An obvious question is whether or not every partial Steiner triple system has an embedding. This question was answered in the affirmative by Treash in 1971 [15]. At this point it is worth remarking that Steiner triple systems are equivalent to idempotent commutative quasigroups in which $x \cdot y = x/y$ (multiplication and division are the same). In [13] it is noted that one motivation for studying embedding problems is that the word problem is solvable for any finitely presented algebra in a variety $\mathcal{V}$ if and only if there is an algorithm for deciding whether a finite partial algebra in $\mathcal{V}$ can be embedded in an algebra in $\mathcal{V}$.

The embeddings constructed by Treash are quite large. The order of the containing Steiner triple system is exponential in the order of the embedded partial system. In 1975, Lindner remedied this situation by proving that any partial Steiner triple system of order u can be embedded in a Steiner triple system of order $6u+3$ [12]. He also made the following conjecture [13].

Conjecture. (Lindner) *Any partial Steiner triple system of order u can be embedded in a Steiner triple system of order v provided v is admissible and $v \geq 2u+1$.*

Obviously some partial Steiner triple systems of order u have embeddings of order $v < 2u+1$, but Lindner's Conjecture is best possible in the following sense. For every $u \geq 9$, there exists a partial Steiner triple system of order u which cannot be embedded in any Steiner triple of order $v < 2u+1$ [7]. To see this, consider a (complete) Steiner triple system $(U, \mathcal{A})$ of admissible order u and suppose we wish to embed it in a Steiner triple system $(U \cup W, \mathcal{B})$ of order $v = u + w$ (where $|W| = w$). Then any $x \in W$ must occur in a triple in $\mathcal{B}$ with each $y \in U$. However, since no two elements of U can occur together in a triple with x (as they already occur in a triple in $\mathcal{A}$) we require $w \geq u+1$ and hence $v \geq 2u+1$. It is easy to see that this lower bound on v remains even when the number of pairs not occurring in the triples of $\mathcal{A}$ is non-zero, provided it is small. Such partial Steiner triple systems are known to exist, see [7].

It is worth mentioning that Colbourn [4] has shown that for some values of $v < 2u+1$, it is NP-complete to decide if an arbitrary partial Steiner triple system of order u can be embedded in a Steiner triple system of order v. This sheds some light on why settling Lindner's Conjecture is not easy.

Prior to the recent resolution of Lindner's Conjecture, considerable work had been done on embeddings of partial Steiner triple systems with many partial results on the conjecture being obtained. In 1980 Andersen et al [1] proved that Lindner's Conjecture holds for a large family of partial Steiner triple systems, and that it holds when $v \geq 4u+1$. This lower bound was recently reduced to $3u-2$ in [2]. In [8] it was shown that a (complete) Steiner triple system of order u can be embedded in a Steiner triple system of order v if and only if $v = u$ or $v \geq 2u+1$ and is admissible, whilst in [5] and [9] embeddings for certain classes of partial Steiner triple systems were constructed. A survey of results on embeddings can be found in [7].

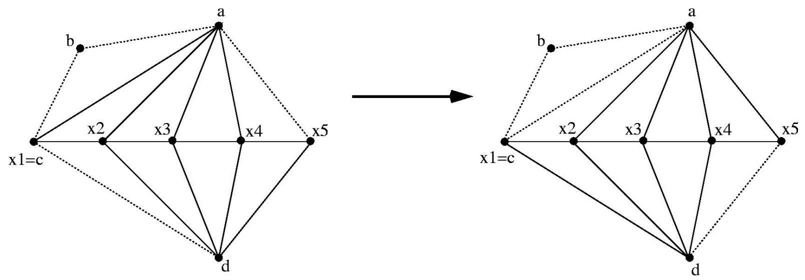
3 A Solution to Lindner's Embedding Problem

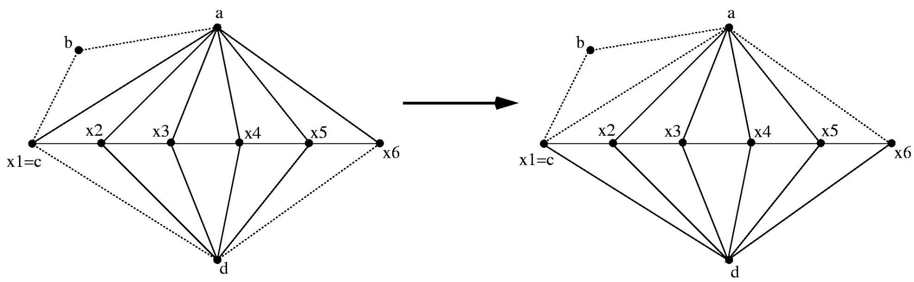
Lindner's embedding problem for partial Steiner triple systems was recently solved [3], and we give here a very brief outline of the solution. The full proof is quite lengthy. An essential ingredient is the notion of *repacking* which we illustrate with a simple example.

Let $(V, \mathcal{B})$ be a partial Steiner triple system of odd order and suppose there is some pair $\{a, b\}$ of elements of V which occurs in no triple of $\mathcal{B}$. We wish to construct a new partial Steiner triple system $(V, \mathcal{B}')$ where $\mathcal{B}'$ has more triples than $\mathcal{B}$. Since the system has odd order, it follows that there is a $c \in V$ such that the pair $\{b, c\}$ occurs in no triple. If there is also no triple containing the pair $\{a, c\}$, then we can add the new triple $\{a, b, c\}$ and we are finished. If not, then it follows (from the fact that the system has odd order) that there is an element $d \in V$, where a, b, c and d are pairwise distinct, such that the pair $\{c, d\}$ occurs in no triple.

Now let $x_1 = c$ and construct the sequence $x_1, x_2, \ldots, x_n$ where for $i = 1, 2, \ldots, n-1$, $\{a, x_i, x_{i+1}\} \in \mathcal{B}$ for i odd, and $\{d, x_i, x_{i+1}\} \in \mathcal{B}$ for i even. This sequence eventually terminates, at x_n, where the pair $\{a, x_n\}$ occurs in no triple (in which case n is odd), or where the pair $\{d, x_n\}$ occurs in no triple (in which case n is even). This is illustrated for the cases $n = 5$ and $n = 6$ at left in the figure below. The triples are represented by solid triangles, whilst dotted lines join pairs occurring in no triple.

If we let $(V, \mathcal{B}^\dagger)$ be the partial Steiner triple system obtained by interchanging a with d in these $n-1$ triples, then we can add the new triple $\{a, b, c\}$ to obtain a new partial Steiner triple system with an extra triple. This procedure, called *path switching* and illustrated in the figure above, is a simple example of repacking and was first used in [1]. In general, repacking involves rearranging or repacking an existing set of triples so that a new system with more triples is obtained. Note that the above path switch only alters triples involving a or d. This means the technique can be used for embedding whenever a and d do not occur in specified triples.





Unfortunately, this path switching method only produces a new triple if $x_n \neq b$. Otherwise embedding partial Steiner triple systems would be much easier. The solution of Lindner's embedding problem is based on several new repacking techniques which involve complicated combinations of path switches. Roughly speaking, these new repacking techniques are used to construct partial Steiner triple systems which are "close" to complete embeddings and have certain desirable properties. The embeddings are then completed via a combination of graph colouring methods and further repacking.

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