On the zeros of finite sums of exponential functions

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Abstract

We establish an upper bound for the number of zeros of finite sums of exponential functions defined in the set of real numbers, and then discuss some applications of this result to linear algebra. Also, we record a few questions for classroom conversations on the modifications of our theorem.
Key words: Finite sum, exponential function, basis of $\mathbb{R}^n$.
MSC: 26A09, 15A03, 15A45.

1 Introduction

Almost every college student knows that a polynomial (in one real variable) of degree n has at most n zeros. Fewer of them notice that a similar result is true for the finite sum of exponential functions. More precisely, the following theorem holds.

Theorem 1 *For $n \geq 1$, let $p_0 > p_1 > \cdots > p_n > 0$ and, for $j = 0, 1, ..., n$, let α_j be a real number so that $\alpha_0 \neq 0$. Then the function $f : \mathbb{R} \to \mathbb{R}$,*

$$f(t) = \sum_{j=0}^{n} \alpha_j p_j^t \tag{1}$$

has at most n zeros.

In other words, the similarity lies in the relationship between the number of terms in the finite sum and the upper bound for the number of zeros; recall that a polynomial of degree n is a finite sum of $n+1$ power functions, possibly with some coefficients equal to zero. We shall give an elementary proof for Theorem 1 that is based upon just a few basic facts in real analysis. Therefore, it is suitable to be presented in classrooms, also at the college level.

In the last section of this paper, we shall also discuss a few applications of Theorem 1 to linear algebra. The purpose of this section is, on the one hand, to demonstrate once again how knowledge in one field of mathematics is often applicable in another field, and on the other hand, to indicate that even elementary methods of classical real analysis are still of use in modern mathematical research.

2 Proof of Theorem 1

We prove the theorem by using mathematical induction and Rolle's theorem. Consider first the case $n = 1$. By writing $q_0 = p_0/p_1$, we have

$$f(t) = p_1^t \Big(\alpha_0 q_0^t + \alpha_1 \Big) = p_1^t g(t),$$

where $f(t) = 0$ if and only if $g(t) = 0$. Since $q_0 > 1$ and $\alpha_0 \neq 0$, there is at most one value $t = t_1$ such that $g(t_1) = 0$.

Assume then that the claim holds for some $n = k \geq 1$ and study the function

$$f(t) = \sum_{j=0}^{k+1} \alpha_j p_j^t.$$

Similarly as above, we denote $q_j = p_j/p_{k+1}$ in order to have $f(t) = p_{k+1}^t g(t)$, where

$$g(t) = \sum_{j=0}^{k} \alpha_j q_j^t + \alpha_{k+1},$$

in which $q_0 > \cdots > q_k > 1$. Again, $f(t) = 0$ if and only if $g(t) = 0$.

Now, for all $t \in \mathbb{R}$, the derivative of g is

$$\begin{aligned} g'(t) &= \sum_{j=0}^{k} \alpha_j (\ln q_j) q_j^t \\ &= \sum_{j=0}^{k} \beta_j q_j^t, \end{aligned}$$

where $\beta_0 \neq 0$. By assumption, at most k distinct numbers $\delta_1 > \cdots > \delta_k$ exist such that $g'(\delta_1) = \ldots = g'(\delta_k) = 0$. Thus, by Rolle's theorem, there are at most $k+1$ distinct numbers $t_1 > \cdots > t_{k+1}$ such that $g(t_1) = \ldots = g(t_{k+1}) = 0$. Theorem 1 follows.

Observe also that Theorem 1 is sharp in the sense that, for every $n \geq 1$, it is possible to construct a function f satisfying the conditions of the theorem so that f has exactly n zeros.

3 Further questions

In this section, we raise a few questions related to Theorem 1 for classroom conversations. In particular, some students might have an interest in considering, for example, the following problems.

a) Let $n = 1$ in Theorem 1. How does the number of zeros of the function f in (1) depend on the values of α_0 and α_1?

b) Do we get an analogy of Theorem 1 if the exponential functions in (1) are replaced with any monotonic real functions?

c) Exponential and logarithm functions share many properties (continuity, monotonicity etc.). Therefore, one might think that a similar result to Theorem 1 for finite sums of logarithm functions would hold (in the set of the positive real numbers) and then try to prove it by replacing every p_j^t with $\log_{p_j} t$, where $p_j > 1$, in the proof of Theorem 1. Unfortunately, this will not work. Why?

d) One can show that the function $g : (0, \infty) \to \mathbb{R}$,

$$g(t) = \sum_{j=0}^{n} \alpha_j \log_{p_j} t,$$

where $p_0 > \cdots > p_n > 1$ and $\alpha_0 \neq 0$, is either strictly monotonic with $g(1) = 0$ or $g(t) = 0$ for every $t > 0$ and that the latter condition occurs if and only if

$$\frac{\alpha_0}{\ln p_0} + ... + \frac{\alpha_n}{\ln p_n} = 0.$$

Which properties of logarithm functions are needed for this result?
e) What can be said of the number of zeros of finite sums of sine or cosine functions?

Clearly, in order to be able to answer these questions, one may have to specify them first; for example, one has to decide to consider in **e)** either the sines of the same parameter with different amplitudes or the sines with different parameters etc.

4 A few applications to linear algebra

There are many problems in mathematics that can be solved by a very elementary method if one is ready to carry out the required, and often a large, number of calculations. However, often a little extra theoretical knowledge might lead us to the final solution more quickly. As an example of this, consider first the trouble when determining whether or not the following set X of vectors in $\mathbb{R}^n$, $n \geq 2$, is linearly independent, by solving the corresponding system of linear equations or using any other elementary method of linear algebra;

$$X = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\}, \tag{2}$$

where $\mathbf{x_j} = (p_j^{t_1} \cdots p_j^{t_n}) \in \mathbb{R}^n$, $p_j > 0$ and $t_j \in \mathbb{R}$ are numbers such that $p_j \neq p_k$ and $t_j \neq t_j$ whenever $j \neq k$.

Using Theorem 1, we shall now see in a simple and quick way that X in (2) is always a basis for $\mathbb{R}^n$. Of course, this is not a major result in modern mathematics but demonstrates once again how expertise in one field of mathematics is often of use in other fields as well. On the other hand, we would like to point out that one may run into this kind of basis candidates for $\mathbb{R}^n$ in several and sometimes quite surprising occasions, for instance, while searching for the upper bounds for the sum of the entries of matrix $\mathbf{A}^m$ in terms of the data of a nonnegative square matrix $\mathbf{A}$. At the end of this section, we shall give some references of this research and shortly tell how Theorem 1 is related to it.

Corollary 1 *For $n \geq 2$ and $j = 1, ..., n$, let $p_j > 0$ and $t_j \in \mathbb{R}$ such that $p_j \neq p_k$ and $t_j \neq t_j$ whenever $j \neq k$. Then $X = \{\mathbf{x_1}, ..., \mathbf{x_n}\}$, where*

$$\mathbf{x_j} = (p_j^{t_1} \cdots p_j^{t_n}),$$

is a basis for $\mathbb{R}^n$.

Proof of Corollary 1. Since the number of vectors in X equals the dimension of $\mathbb{R}^n$, it is enough to show that

$$\sum_{i=1}^{n} \alpha_i \mathbf{x_i} = \mathbf{0} \quad \text{implies} \quad \alpha_1 = \cdots = \alpha_n = 0.$$

Also, because n is finite, we may without loss of generality suppose that $p_1 > \cdots > p_n$, cf. how the value of an $n \times n$ determinant is affected when two rows of it are interchanged.

Suppose that $\sum_{i=1}^{n} \alpha_i \mathbf{x_i} = \mathbf{0}$. Now

$$\sum_{i=1}^{n} \alpha_i \mathbf{x_i} = \mathbf{0} \Leftrightarrow f(t_1) = \cdots = f(t_n) = 0,$$

where f is the function in (1) (with the obvious modification $j = i-1$ in (1)). Hence we must have $\alpha_1 = 0$ by Theorem 1. Repeating a similar argument for the partial sums $\sum_{i=k}^{n} \alpha_i \mathbf{x_i}$, $k = 2, ..., n-1$, we are led to the conclusion that $\alpha_2 = \cdots = \alpha_n = 0$ as well. The corollary follows. □

Finally, we observe that, by letting $t \to \infty$, we also deduce the following consequence for Theorem 1.

Corollary 2 *Let f be the function of Theorem 1 with $\alpha_0 > 0$. If*

$$f(t_1) = \cdots = f(t_n) = 0$$

for real numbers $t_1 > \cdots > t_n$, then $f(t) > 0$ for every $t > t_1$.

This fact, although a simple one, was one of the key elements to establish the most recent results in a larger study, e.g. [2], [3] and [5], on the upper bounds for the sum of the entries of matrix $\mathbf{A}^m$, $m \geq 2$, in terms of the column and row sums of $n \times n$ matrix $\mathbf{A}$. The detailed description of these results is given in [2].

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Received 26 July 2005, accepted for publication 14 September 2005.