A class of applications of AM-GM Inequality:
From a 2004 Putnam competition problem to Lalescu's sequence

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Discussing a series of inequalities, B. Bollobás reminds us in [3] that Harald Bohr wrote: "All analysts spend half their time hunting through the literature for inequalities which they want to use but cannot prove." Fortunately, other inequalities can be reduced to techniques whose strategy of proof is familiar to us. This expository note has been inspired by the problem B2 from the 2004 edition of the W.L. Putnam competition. We will show that a natural context where this problem can be discussed is the area of applications of arithmetic mean-geometric mean (AM-GM) inequality. We conclude our note with a presentation of a classic problem, the Lalescu's sequence.

Using only elementary arguments, we can show that the sequence $x_n = \left(1+\frac{1}{n}\right)^n$ is increasing and convergent to a limit between 2 and 3 that we denote by e. The fact that x_n is increasing is mentioned in [5], p. 37, application 35. In [7], this fact is proved as an application of the AM-GM inequality.

To remind ourselves here of the proof that $\{x_n\}_{n\in\mathbb{N}}$ is increasing, we start with the AM-GM inequality. For $a_1, a_2, ..., a_n \geq 0$, we have:

$$\frac{a_1+a_2+...+a_n}{n} \geq (a_1a_2...a_n)^{1/n},$$

with equality if and only if $a_1 = a_2 = ... = a_n$. This inequality has many simple proofs. For example, a proof based on the concavity of the logarithmic function is presented in various sources, and the original reference is Jensen's paper [6]. A proof based on induction, given by Cauchy in 1821, is presented in many sources, as for example in [3], pp. 1–2. We use the AM-GM inequality to show that $\{x_n\}_{n\in\mathbb{N}}$ is increasing. We apply this inequality to $n+1$ positive real numbers, $a_1 = 1$, $a_2 = ... = a_{n+1} = 1+\frac{1}{n}$. We get:

$$\frac{1+n(1+\frac{1}{n})}{n+1} > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}},$$

which yields immediately:

$$x_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n = x_n.$$

Thus, we proved that $\{x_n\}_{n\in\mathbb{N}}$ is increasing.

It is well known that this fact can be proved also by using the Bernoulli inequality (see, for example, [8], vol. 1, p. 63):

$$(1+t)^n > 1+nt, \quad t \in (-1,\infty) - \{0\},$$

by computing the ratio

$$\frac{x_{n+1}}{x_n} = \left(\frac{n+2}{n+1}\right)^{n+1} : \left(\left(\frac{n+1}{n}\right)^{n+1} \cdot \frac{n}{n+1}\right) = \frac{n+1}{n} \cdot \left(\frac{n^2+2n}{(n+1)^2}\right)^{n+1} =$$

$$= \left(1-\frac{1}{(n+1)^2}\right)^{n+1} \cdot \frac{n+1}{n} > \left(1-\frac{n+1}{(n+1)^2}\right) \cdot \frac{n+1}{n} = 1.$$

Application 1 (Problem B2, the Sixty-Fifth W.L. Putnam Mathematical Competition, December 4, 2004.) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}.$$

Solution: Let m be fixed. We prove the statement by induction for $n \geq 1$. For $n = 1$, the inequality we need to prove is

$$\frac{(m+1)!}{(m+1)^{m+1}} < \frac{m!}{m^m}.$$

Simplifying, we get:

$$(m+1) < \frac{(m+1)^{m+1}}{m^m} = (m+1) \cdot \frac{m+1}{m} \cdots \frac{m+1}{m},$$

which is obviously true.

Suppose now the statement is true for n, and let's prove it for $n+1$. We need to prove the following statement:

$$\frac{(m+n+1)!}{(m+n+1)^{m+n+1}} < \frac{m!}{m^m} \cdot \frac{(n+1)!}{(n+1)^{n+1}}.$$

We start with the right hand side term. We use the induction hypothesis:

$$\frac{m!}{m^m} \cdot \frac{(n+1)!}{(n+1)^{n+1}} = \frac{m!}{m^m} \cdot \frac{(n+1) \cdot n!}{(n+1)^{n+1}} \cdot \frac{n^n}{n^n} > \frac{(m+n)!}{(m+n)^{m+n}} \cdot \frac{(n+1) \cdot n^n}{(n+1)^{n+1}}.$$

Now it suffices to prove:

$$\frac{(m+n)!}{(m+n)^{m+n}} \cdot \frac{(n+1) \cdot n^n}{(n+1)^{n+1}} > \frac{(m+n+1)!}{(m+n+1)^{m+n+1}}.$$

This last inequality is true if and only if the following inequality is true:

$$n^n \cdot (m+n+1)^{m+n} > (n+1)^n \cdot (m+n)^{m+n}$$

which reduces to

$$x_{m+n} = \left(\frac{m+n+1}{m+n}\right)^{m+n} > \left(\frac{n+1}{n}\right)^n = x_n.$$

This is true, as we proved above, since the sequence $\{x_n\}$ is increasing.

Note Alternative solutions to the problems from the 2004 edition of the W.L. Putnam Competition have been published in *Mathematics Magazine* **78** (2005), 76–80, and in *American Mathematical Monthly* **112** (2005), 713–725. However, the solution we presented above for Problem B2 is different from these published solutions.

Remark This inequality can be iterated to obtain the following extension. Let $m_1, ..., m_k$ be positive integers, for $k \geq 2$. Then

$$\frac{(m_1 + ... + m_k)!}{(m_1 + ... + m_k)^{m_1 + ... + m_k}} < \frac{m_1!}{m_1^{m_1}} \cdots \frac{m_k!}{{m_k}^{m_k}}.$$

A problem where the AM-GM inequality is applied on the terms of the sequence $x_n = \left(1 + \frac{1}{n}\right)^n$ has been assigned previously in the Putnam competition, as we see in the following example.

Application 2 (The Thirty-Sixth W.L. Putnam Mathematical Competition, Dec. 6, 1975) Prove that if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$, then

(a) $n(n+1)^{1/n} < n + s_n$, for $n > 1$, and

(b) $(n-1)n^{-1/(n-1)} < n - s_n$, for $n > 2$.
Solution: We present here the solution from [1], p.94. Applying the AM-GM inequality, we have:

$$\frac{n+s_n}{n} = \frac{(1+1)+(1+\frac{1}{2})+...+(1+\frac{1}{n})}{n} >$$

$$> \left[(1+1)(1+\frac{1}{2})\cdots(1+\frac{1}{n})\right]^{1/n} = \left(2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{(n+1)}{n}\right)^{\frac{1}{n}} = (n+1)^{\frac{1}{n}},$$

and this proves part (a). For the proof of part (b), we see that:

$$\frac{n-s_n}{n-1} = \frac{(1-\frac{1}{2})+...+(1-\frac{1}{n})}{n-1} >$$

$$> \left[(1-\frac{1}{2})\cdot...\cdot(1-\frac{1}{n})\right]^{\frac{1}{n-1}} = \left(\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{(n-1)}{n}\right)^{\frac{1}{n-1}} = n^{-\frac{1}{n-1}}.$$

A classical exercise from mathematical folklore is the following application, where we will use the fact that $x_n = \left(1+\frac{1}{n}\right)^n < e$. (See, for example, [8], p.44.)

Application 3 Prove that

$$\lim_{n\to\infty} \frac{1^1+2^2+...+n^n}{(n!)^2} = 0.$$

Solution: We have the obvious inequality

$$\alpha_n = \frac{1^1+2^2+...+n^n}{(n!)^2} < \frac{n\cdot n^n}{(n!)^2} = \beta_n,$$

for all integers $n \geq 1$. If we prove that $\lim_{n\to\infty}\beta_n = 0$, we are done. Too see this, we show first that the sequence $\{\beta_n\}$ is decreasing:

$$\frac{\beta_{n+1}}{\beta_n} = \frac{(n+1)^{n+2}}{((n+1)!)^2}\cdot\frac{(n!)^2}{n^{n+1}} = \frac{1}{n}\left(1+\frac{1}{n}\right)^n < \frac{1}{n}\cdot e.$$

The ratio $\frac{e}{n}$ is less than 1 for all $n \geq 3$, which proves that β_n is a decreasing sequence. To find the limit, let us remark first that this sequence satisfies the recurrence relation

$$\beta_{n+1} = \frac{1}{n}\left(1+\frac{1}{n}\right)^n \beta_n,$$

thus $\lim_{n\to\infty}\beta_n = 0$. Since $0 < \alpha_n < \beta_n$ for all $n \geq 1$, we have $\lim_{n\to\infty}\alpha_n = 0$.

The fact that $x_n = \left(1+\frac{1}{n}\right)^n$ is increasing and converges to e suggests also the study of the auxiliary function $f : (0,\infty) \to \mathbb{R}$, $f(x) = (1+\frac{1}{x})^x$. Of course, the differential approach can be useful in some applications, as it is, for example, the following problem.

Application 4 Find all positive integers such that

$$(n+3)^n = \sum_{k=3}^{n+2} k^n.$$

The solution is presented in [2], pp. 56–57; this problem has been proposed in a national contest in France, in 1999. The answer is that $n = 2$ and $n = 3$ are the only solutions to the equation. The solution uses, besides the derivative of the function $\log f$, the Taylor expansion about $a = 0$ of the function $\log(1+x)$.

In conclusion, we can say that the problem B2 from the 2004 edition of the Putnam Competition is related to a class of problems where the properties of the function $f(x) = \left(1+\frac{1}{x}\right)^x$ are studied. The general context to discuss these properties, as for example [5] and

[7] suggest it, is the class of applications of AM-GM inequality. This class of applications are related to the study of Lalescu's sequence, as we'll see in the next part of our note.

We continue our presentation with the following exploration. We have seen that $x_n = \left(1+\frac{1}{n}\right)^n$ is increasing. We try to produce an inequality similar to the one in Application 1, starting from the fact that $m < n$ implies

$$\left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{m}\right)^m.$$

We rewrite this inequality as

$$\frac{(n+1)^n}{n^n} > \frac{(m+1)^m}{m^m}.$$

Cross-multiplying and dividing both sides by $n!m!$, we get:

$$\frac{m^m}{m!} \cdot \frac{(n+1)^n}{n!} > \frac{n^n}{n!} \cdot \frac{(m+1)^m}{m!}$$

which can be rewritten as

$$\frac{m^m}{m!} \cdot \frac{(n+1)^{n+1}}{(n+1)!} > \frac{n^n}{n!} \cdot \frac{(m+1)^{m+1}}{(m+1)!}.$$

This is the inequality we were looking for. Now we can summarize these computations.

Application 5 Let $\theta_n = \frac{n^n}{n!}$. Then the following statements are true:

(a) If $m < n$, then $\theta_m \cdot \theta_{n+1} > \theta_n \cdot \theta_{m+1}$;

(b) $\theta_{m+n} < \theta_m \cdot \theta_n$;

(c) $\lim_{n\to\infty} \frac{\theta_n}{(n-1)!} = 0$.

We have proved (a) in the remarks preceding the statement of Application 5. Application 1 is (b), and we have used (c) as an argument in the proof of Application 3.

Application 6 Denote $z_n = \left(\frac{n!}{n^n}\right)^{1/n} = \theta_n^{-1/n}$. Then

$$\lim_{n\to\infty} z_n = \frac{1}{e}.$$

Solution: Since $\lim_{n\to\infty} \frac{\theta_n}{\theta_{n+1}}$ exists and it is not 0, we have

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (\theta_n)^{-1/n} = \lim_{n\to\infty} \frac{\theta_n}{\theta_{n+1}} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} =$$

$$\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}.$$

Finally, we will discuss here the famous Lalescu's sequence, known since the beginning of the 20th century.

Application 7 Prove that

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \left(((n+1)!)^{1/n+1} - (n!)^{1/n} \right) = \frac{1}{e}.$$

We present here two proofs of this application. To establish notation, we will denote the above limit by L.

Solution 1: (Due to Marcel Ţena, see [8], vol. 2, pp. 93.) First, remark that

$$\lim_{n\to\infty} \frac{[(n+1)!]^{1/n+1}}{(n!)^{1/n}} = \lim_{n\to\infty} \frac{[(n+1)!]^{1/n+1}}{n+1} \cdot \frac{n+1}{n} \cdot \frac{n}{(n!)^{1/n}} = 1,$$

where for the last equality we have used Application 6.

Now we compute

$$\lim_{n\to\infty}\left(\frac{((n+1)!)^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}}\right)^n$$

in two different ways.

First, let

$$b_n = \frac{[(n+1)!]^n}{(n!)^{n+1}}$$

and compute

$$\lim_{n\to\infty}\frac{b_n}{b_{n-1}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e.$$

Secondly, using the notation $L_n = ((n+1)!)^{1/n+1} - (n!)^{1/n}$, we have:

$$\lim_{n\to\infty}\left(\frac{[(n+1)!]^{1/n+1}}{(n!)^{1/n}}\right)^n =$$

$$\lim_{n\to\infty}\left[\left(1+\frac{((n+1)!)^{1/n+1}-(n!)^{1/n}}{(n!)^{1/n}}\right)^{\frac{(n!)^{1/n}}{((n+1)!)^{1/n+1}-(n!)^{1/n}}}\right]^{\frac{n}{(n!)^{1/n}}\cdot L_n}$$

Thus, in the second way of computing the limit we get $e^{e\cdot\lim L_n}$. Comparing the two different ways of getting the same limit, we obtain that $e = e^{e\cdot\lim L_n}$. Thus, $\lim_{n\to\infty} L_n = \frac{1}{e}$.

Solution 2: As far as we know, this solution appears here for the first time. Observe that L_n may be written in the form

$$L_n = \frac{(n!)^{1/n}}{n}(A_n - 1)n,$$

where $A_n = \frac{(n+1)!^{1/n+1}}{(n!)^{1/n}}$. Then, we can rewrite

$$L_n = \frac{(n!)^{1/n}}{n}\left(\frac{e^{\log A_n}-1}{\log A_n}\right)\cdot n\log A_n.$$

On the other hand

$$\log A_n = \frac{1}{n+1}\sum_{k=1}^{n+1}\log k - \frac{1}{n}\sum_{k=1}^{n}\log k = \frac{1}{n+1}\log(n+1) - \frac{1}{n(n+1)}\sum_{k=1}^{n}\log k,$$

which means

$$\log A_n = \frac{1}{n(n+1)}\log\frac{(n+1)^n}{n!}.$$

Therefore,

$$L_n = \frac{(n!)^{1/n}}{n}\cdot\frac{e^{\log A_n}-1}{\log A_n}\cdot\left[\log\left(\frac{(n+1)^n}{(n+1)!}\right)^{\frac{1}{n+1}} + \log(n+1)^{\frac{1}{n+1}}\right].$$

In this last relation, using Application 6, $\lim\log A_n = 0$, and $\lim n^{\frac{1}{n}} = 1$, we get that $\lim L_n = \frac{1}{e}$.

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