“Mathematics” vs Mathematics

I. Bokor

1 Introduction

Australia is experiencing a crisis in science in general, and in mathematics in particular. It is cold comfort that Australia is hardly alone.

The fact that so many colleagues from other countries consistently report similar problems not only makes the situation of even greater concern, but also suggests that the Australian experience could be instructive to other communities, especially since Australia seems to be further down this perilous track.

This crisis did not befall Australia suddenly. Rather, it is the culmination of developments spanning more than two decades. The academic community has been alert to this crisis for a while, but apparently saw itself as powerless to affect the situation.

It is to the credit of the Federation of Australian Technological and Scientific Societies (FASTS) that it has contributed so much to parliamentarians and members of the public becoming aware of both the gravity of the situation and the urgency of the need for action.

Resolving the crisis is more than a matter of increasing public and parliamentary awareness, as well as soliciting financial and moral support. There is also a number of things the academic community could do, that do not require significant increases in funding.

One obvious one is to re-establish the severed link between science and mathematics on the one hand, and the teaching of science and the teaching of mathematics on the other. Whereas a quarter of a century ago high school teachers were expected to have a degree in the discipline(s) they taught, together with teaching qualifications, it is all too common today to find teachers with only “education” qualifications.

There are numerous consequences of this severance. Two related ones are that less and less can be assumed to be familiar to students leaving high school, and that more and more “foundation courses” — or whatever the current euphemism might be for remedial courses — are needed and offered at universities.

The view seems to be commonly held, even amongst mathematics and science educationists, that mere passing familiarity with rudimentary facts is more than adequate for teaching these subjects at school. School teachers, and even those who train them, frequently argue that it is actually preferable to know less about mathematics and science in order to teach them better.

There is a confusion of numeracy with mathematics, a fallacy as crude as equating literacy with literature. For while one must be numerate in order to attempt to learn or appreciate mathematics, there is a *qualitative* difference between mathematics and mere computation, just as a narrative text does not become literature solely because it is free of grammatical errors and spelling mistakes.

Partly as a consequence, there is a pervasive belief, including among those who use and apply mathematics, that any mathematical problem has a unique solution, which can be readily computed numerically if one just had the right computer with the right programme, or, failing that, by being adept enough.

But even elementary arithmetic naturally provides ample opportunities for exposing such pernicious nonsense as just that, at the same time as providing a convenient path to lead a pupil or student from readily accessible problems to increasingly sophisticated and powerful theories and techniques. It is the use, as opposed to the neglect or abuse, of these opportunities which marks the difference between mathematics and “mathematics”.

2 A Sample Problem

Let us examine one such opportunity, namely the one offered by

Problem 1 $1, 2, 3, x, \ldots$ *Find x.*

When this problem is posed to students or school teachers, the typical first response is a derisory laugh. When pressed, the audience’s most common reply is that the problem is trivial, for the answer is “obviously 4”. When pressed even further for an explanation, the most common one offered is that this is just the way we count.

This is plainly a very reasonable explanation for suggesting that x is 4, especially in light of the fact that when we teach numbers and counting to young children, we inculcate that 4 comes immediately after 3, just as when we begin to teach reading and writing, we inculcate that d is the next letter after c in the English alphabet.

We properly consider it an indication of success at learning when a young child says “4” in answer to Problem 1, and would usually correct any other answer.

In “mathematics”, that settles the matter once and for all, and one simply “moves on”, wondering rhetorically what the fuss was all about, questioning the seriousness or sanity of any person wishing to pursue the matter further.

In mathematics, on the other hand, it is only the beginning of the matter.

Let us look at the problem a little more closely, even at the risk of ridicule, because variants of Problem 1 are actually encountered quite frequently, in manifold guises.

One obvious place is in *intelligence tests* where a string of numbers is given and the subject required to find “the next number” fitting the pattern established by the given ones.

The fact that many significant decisions, with far-reaching consequences, are made on the basis of such “psychometric tests” is sufficient ground for deeper investigation.

However confident psychometricians might be, even a superficial analysis of the problem reveals that caution is warranted. For as the following examples clearly illustrate, while it might be “obvious” that “the” answer is 4, this is not necessarily true.

Example 1 It is Saturday night and the Lotto numbers are being drawn. The first three drawn are 1,2,3. Quite a few people would like to know x — and the subsequent numbers!

Example 2 In a three-handed game, cards are dealt one at a time, with players receiving their cards in order. Thus after Player 1 is dealt a card, Player 2 receives one, followed by Player 3, after whom it is again Player 1’s turn to receive a card. So if our “$1, 2, 3$” is what the dealer says to himself when dealing the cards to Players 1, 2 and 3 in order, “x” will be 1 until the dealing has been completed, and then there will be no “x”.

Example 3 A pupil has just heard of *prime numbers*, and wishes to list all the counting numbers which have no divisors other than themselves and 1. Clearly “x” is 5 in this case.

Example 4 The complex numbers and the quaternions play an important rôle in mathematics and physics. They each form an algebraic structure known as *division algebras* over the real numbers. There is a similar algebraic object called the *octonians*, for which, however, the "multiplication" fails to be associative. These have dimension 2, 4 and 8, respectively, over the real numbers. From the nineteenth century, there was an attempt to find all other similar non-trivial algebraic structures. By the middle of the twentieth century, it was known that the answer to this essentially arithmetic problem depended on the *Hopf Invariant 1* problem of algebraic topology. It was also known that the dimension of any such object must be a power of 2: $2 = 2^1$, $4 = 2^2$, $8 = 2^3$. Moreover, the examples listed were the only ones known at the time. Thus we may regard "$1, 2, 3, x, \ldots$ Find x." as a formulation of the Hopf Invariant 1 problem:

> Find the next power of 2 which is the dimension of a (not necessarily associative) division algebra over the reals.

J.F. Adams showed in 1960, in effect, that there is no x.

These examples force the conclusion that, as it stands, Problem 1 is ill-formed, and therefore *cannot* have a solution.

This is a very common phenomenon to anyone working in mathematics or any of the sciences. We are frequently confronted with problems which are ill-formed. It is often necessary to supplement the given information. The above examples demonstrate that it is this supplementary information — in other words, additional assumptions — that *determine* what comes into question as a possible solution.

The above examples illustrate that Problem 1 can have

(1) no solutions, as in Example 4,
(2) a unique solution, as in Example 3, or
(3) multiple solutions, as in Example 1.

Thus we confront one of the two principal difficulties with mathematics:

> *The obvious is not always true.*

Of course, none of these is likely to be what someone posing Problem 1 "has in mind". Much more likely is the tacit assumption that "$1, 2, 3$" is intended to be the beginning of a sequence of (whole) numbers determined by a "definite rule". Moreover, the expected "obvious", or "natural" or "simplest" answer is that each term is one more than the previous term. In other words, the respondent is required to recognise counting as being *the* pattern, rather than being *one* pattern, amongst possibly many, to fit the data.

Such an additional assumption has *everything* to do with the demand to conform to standard ways of perceiving, thinking and behaving, and *nothing* to do with mathematics. It is a paradigm of "mathematics" as opposed to mathematics. It is misinformation passed off as truth and enlightenment.

An example from the study of elasticity readily shows that there are naturally occurring, practical situations in which the first three terms of a sequence of real numbers are as given, the sequence is determined by a fixed and definite rule, and the natural — in fact only! — answer is almost certainly not 4.

Example 5 Hooke's Law states that the tension is proportional to the extension of a solid body, such as a length of wire. This is known to hold with sufficient accuracy for "sufficiently small" extensions. But it breaks down when the extension or tension is "sufficiently large". How small or large is "sufficient" depends on the material composition of the solid body. Thus, choosing the units of measurement appropriately, it could easily happen that applying 1 unit of tension results in 1 unit of extension, 2 in 2, 3 in 3. If the critical value, at which Hooke's Law breaks down, lies between 3 and 4 units, then applying 4 units of tension will *not*, in general, result in 4 units of extension: the natural answer in this case, is almost certainly not 4.

So let us formulate our problem in a form general enough to incorporate our examples while accommodating those who might think that we have taken a "simple","very basic" thing and turned it into an unrecognisable, complicated mess.

Let us regard 1, 2 and 3 as the values taken by a real valued function of a single real variable at the real numbers 0,1 and 2 respectively.

This allows us to reformulate Problem 1 as

Problem 2 *Let X be a set of real numbers containing 0, 1, 2 and 3. Given a function*

$$f : X \longrightarrow \mathbb{R} \tag{1}$$

such that $f(0) = 1, f(1) = 2$ and $f(2) = 3$, find $f(3)$.

Before attempting a solution, let us see how this incorporates our examples.

Example 1

There are six numbers drawn from 1 to 45 and two "supplementary" ones. We label the first number drawn x_1, the second x_2, ... and the eighth x_8. Then $X := \{0, 1, 3, 4, 5, 6, 7\}$ and f is given by

$$f(j) := x_{j+1}. \tag{2}$$

Example 2

Let us suppose that there are $3n$ cards to be dealt, so that each player receives n cards. We take $X := \{0, 1, 3, \dots, n-1\}$ and then f is defined by

$$f(j) := \begin{cases} 1 & \text{if } j \equiv 0 \mod 3 \\ 2 & \text{if } j \equiv 1 \mod 3 \\ 3 & \text{if } j \equiv 2 \mod 3 \end{cases} \tag{3}$$

Example 3

Writing p_j for the j^{th} prime number, we take $X := \mathbb{N}$, the set of all natural numbers, and f given by

$$f(j) := \begin{cases} 1 & \text{if } j = 0 \\ p_j & \text{if } j > 0 \end{cases} \tag{4}$$

Example 4

Here we take $X := \{0, 1, 2\}$ and f given by

$$f(j) = 2^{j+1} \tag{5}$$

Example 5

The situation is more complicated here, for it is not possible to give an explicit function. It is the subject of extensive experimentation to find for each metal or alloy the function which describes its behaviour under given ambient conditions. The set of values for which the function is linear also varies from material to material. Those who have experience with experimental physics or materials engineering will have had ample exercise in approximating the function in question.

We have just illustrated how the mathematical notion of a function can be exploited to model certain processes and procedures. The functions which arose varied quite significantly in sophistication, and so the objection could be raised that they are "too fancy" for our original purposes.

So let us restrict attention to computationally "simple" solutions — ones expressible in terms of polynomials. We then reformulate Problem 1 as

Problem 3 *Suppose that the polynomial $p(t)$ satisfies $p(0) = 1, p(1) = 2$ and $p(2) = 3$. Find $p(3)$.*

We show that we given any real number x, we can find infinitely many polynomials satisfying $p(0) = 1, p(1) = 2, p(2) = 3$ and $p(4) = x$.

Solution 1 Let $p(t)$ be the polynomial

$$\frac{(t-1)(t-2)(t-3)}{-6} + \frac{2t(t-2)(t-3)}{2} + \frac{3t(t-1)(t-3)}{-2} + \frac{xt(t-1)(t-2)}{6}. \tag{6}$$

Simple direct substitution verifies this polynomial satisfies the prescribed conditions.

Now let $q(t)$ be any polynomial whatsoever. By inspection, the polynomial $r(t)$ defined by

$$r(t) := p(t) + t(t-1)(t-2)(t-3)q(t) \tag{7}$$

satisfies the prescribed conditions, showing that there are infinitely many solutions.

We have used *interpolation* to fit functions of a specified form to given data.

We consider the data to be of the form $\varphi(i) = x_i \quad (i = 0, \ldots 3)$ and the problem is to determine $\varphi : \mathbb{R} \to \mathbb{R}$.

The mathematical idea used is to find functions $f_0, f_1, f_2, f_3 : \mathbb{R} \longrightarrow \mathbb{R}$, with

$$f_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

as well as functions $g_0, g_1, g_2, g_3 : \mathbb{R} \longrightarrow \mathbb{R}$ with $g_j(j) \neq 0$. Then, given any function $h : \mathbb{R} \longrightarrow \mathbb{R}$ whatsoever, the function

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R},\ t \longmapsto \sum_{j=0}^{3} \frac{x_j}{g_j(j)} f_j(t) g_j(t) + t(t-1)(t-2)(t-3)h(t) \tag{9}$$

satisfies our requirements. There are, plainly, infinitely many different solutions.

This method can be applied when n values x_j $(j = 1, \ldots, n)$ are prescribed at n points t_j $(j = 1, \ldots, n)$ in the domain of φ. The solution for the problem arrived at by the above method is then

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R},\ t \longmapsto \sum_{j=1}^{n} \frac{x_j}{g_j(j)} f_j(t) g_j(t) + \prod_{j=1}^{n} (t - t_j) h(t). \tag{10}$$

Solution 2 Another approach to Problem 1 is to propose a polynomial of degree 3, say $p(t) = a + bt + ct^2 + dt^3$, and require it to satisfy $p(0) = 1, p(1) = 2, p(2) = 3$ and $p(4) = x$. These requirements translate into a system of four linear equations in four unknowns:

$$\begin{array}{ccccccccc} a & & & & & & & = & 1 \\ a & + & b & + & c & + & d & = & 2 \\ a & + & 2b & + & 4c & + & 8d & = & 3 \\ a & + & 3b & + & 9c & + & 27d & = & x \end{array} \tag{11}$$

This system can be expressed in terms of matrices.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ x \end{bmatrix}. \tag{12}$$

It is a standard result from elementary linear algebra that this matrix equation has a unique solution, whence we have a uniquely determined polynomial of degree at most 3, $p(t)$, fulfilling our requirements.

In our case, we can *eliminate variables* from the system of equations, or, equivalently, use *elementary row operations* on the *augmented matrix* for the system of equations, to see that the above system is equivalent to the following one, in the sense that any solution of one is also a solution of the other,

$$\begin{aligned} a &= 1 \\ b + c + d &= 1 \\ c + 3d &= 0 \\ 6d &= x - 4. \end{aligned} \tag{13}$$

In terms of matrices, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ x-4 \end{bmatrix}. \tag{14}$$

We now see that for each x there is a unique solution, namely,

$$a = 1, \quad b = \frac{x-1}{3}, \quad c = \frac{-x+4}{2}, \quad d = \frac{x-4}{6}, \tag{15}$$

so that

$$p(t) = \frac{x-4}{6}t^3 - \frac{x-4}{2}t^2 + \frac{x-1}{3}t + 1, \tag{16}$$

as can be readily verified by direct substitution.

It is also an immediate consequence of elementary linear algebra that there are infinitely many polynomials of higher degree satisfying the same conditions. One way to see this is to take any non-zero polynomial $q(t)$ whatsoever and construct the polynomial

$$r(t) := p(t) + t(t-1)(t-2)(t-3)q(t). \tag{17}$$

It is an instructive exercise to show that the polynomial $p(t)$ in Equation 6 and the polynomial $p(t)$ in Equation 16 are, in fact, one and the same polynomial.

Another approach to Problem 1 is to regard $1, 2, 3$ as the first three terms in a sequence of numbers, for definiteness, say real numbers. If we abbreviate the sequence $u_1, u_2, u_3, \ldots$ to $(u_n)_{n\in\mathbb{N}}$, we can reformulate Problem 1 as

Problem 4 *Given a sequence of real numbers, $(u_n)_{n\in\mathbb{N}}$ with $u_0 = 1$, $u_1 = 2$ and $u_2 = 3$, what is u_3?*

Of course any real number x is possible, for it could be that the sequence is the output of a random number generator (cf. Example 1). But even supposing that the sequence is not randomly generated, but is given by some definite rule, does not alter this fact.

One way of seeing this is to use the theory of *difference equations*.

Solution 3 We define $\Delta_n^{(1)} := u_{n+1} - u_n$, and whenever $\left(\Delta_n^{(j)}\right)_{n\in\mathbb{N}}$ has been defined for some $j \in \mathbb{N}$, we define $\Delta_n^{(j+1)} := \Delta_{n+1}^{(j)} - \Delta_n^{(j)}$. For notational uniformity, we write $\Delta_n^{(0)}$ for u_n.

Thus, if $u_0 = 1, u_1 = 2, u_2 = 3$ and $u_3 = x$, we obtain

$$\begin{array}{lccccccc} \Delta_n^{(0)}: & 1 & & 2 & & 3 & & x \\ \Delta_n^{(1)}: & & 1 & & 1 & & x-3 & \\ \Delta_n^{(2)}: & & & 0 & & x-4 & & \\ \Delta_n^{(3)}: & & & & x-4 & & & \end{array} \tag{18}$$

where we have written the difference between successive terms in each row between them in the row immediately below.

Of course, we can use the fact that $\Delta_{n+1}^{(j)} := \Delta_n^{(j)} + \Delta_n^{(j+1)}$ to reconstruct $(u_n)_{n\in\mathbb{N}}$ from the $\Delta_n^{(j)}$ $j, n = 1, 2\dots$ Our data are

$$\Delta_n^{(3)} = x - 4 \qquad \text{for all } n \in \mathbb{N} \tag{19}$$

and

$$\Delta_0^{(2)} = 0, \quad \Delta_0^{(1)} = 1 \quad \text{and} \quad \Delta_0^{(0)} = 1 \tag{20}$$

then we obtain, successively,

$$\begin{array}{lllll} \Delta_0^{(2)} = 0, & \Delta_1^{(2)} = x-4, & \Delta_2^{(2)} = 2x-8, & \dots & \\ \Delta_0^{(1)} = 1, & \Delta_1^{(1)} = 1, & \Delta_2^{(1)} = x-3, & \Delta_3^{(1)} = 3x-11, & \dots \\ \Delta_0^{(0)} = 1, & \Delta_1^{(0)} = 2, & \Delta_2^{(0)} = 3, & \Delta_3^{(0)} = x, & \Delta_4^{(0)} = 4x-11, \end{array} \tag{21}$$

and our sequence $(u_n)_{n\in\mathbb{N}}$ is now completely determined, since $u_n = \Delta_n^{(0)}$.

We have exhibited a *recursive procedure* for constructing a sequence for each choice of x. Our method may seem to be *ad hoc*, but it depends only on the fact that, given k numbers in a given order, then after at most $k-1$ *iterations* of taking differences between successive term, we are left with a constant sequence.

If we express this last sequence of iterated differences using the given terms of the original sequence, we arrive at a *difference equation.* For

$$\begin{aligned} \Delta_n^2 :=& \Delta_{n+1}^1 - \Delta_n^1 \\ =& (u_{n+2} - u_{n+1}) - (u_{n+1} - u_n) \\ =& u_{n+2} - 2u_{n+1} + u_n. \end{aligned} \tag{22}$$

The corresponding calculation for Δ_n^3 allows us to rewrite Equation 19 and thus obtain the *difference equation*

$$u_{n+3} - 3u_{n+2} + 3u_{n+1} - u_n = x - 4 \tag{23}$$

together with *boundary* or *initial conditions*

$$u_0 = 1, \quad \Delta_0^{(1)} = 1, \quad \Delta_0^{(2)} = 0, \tag{24}$$

or, equivalently in this case,

$$u_0 = 1, \quad u_1 = 2, \quad u_2 = 3 \tag{25}$$

and these completely determine $(u_n)_{n\in\mathbb{N}}$. In fact, the general solution of the difference equation 23 is

$$u_n = \frac{x-4}{6}n^3 + Bn^2 + Cn + D \qquad (B, C, D \in \mathbb{R}). \tag{26}$$

Imposing the boundary conditions leads to

$$B = \frac{4-x}{2}, \quad C = \frac{x-1}{3} \quad \text{and} \quad D = 1, \tag{27}$$

so that

$$u_n = \frac{x-4}{6}n^3 - \frac{x-4}{2}n^2 + \frac{x-1}{3}n + 1 \tag{28}$$

3 Observation

We would like to emphasise that we have not rejected the usual immediate response "4". Quite the contrary, this solution plays a distinguished rôle in each of our proposed solutions: When $x = 4$, $p(t)$ in Equation 16 reduces to $t + 1$ and u_n in Equation 28 becomes $n + 1$, both of which produce $1, 2, 3, 4, 5, \ldots$, the sequence of counting numbers. Moreover, it is only when $x = 4$ that $p(t)$, and the polynomial expressing u_n in terms of n, have degree less than 3.

Thus we have neither lost sight of nor rejected our "intuition". We have merely clarified it by placing it into its proper, mathematical context, which, in turn, has led us naturally to mathematically interesting questions:

(1) **Does every similar problem have a distinguished solution?**
(2) **When there is a distinguished solution, is it unique?**

4 Discussion

This is all well and good for those interested in mathematics. But what is there in all of this for those who view mathematics as, at best, a necessary evil, while harbouring pressing doubts about its necessity?

Hopefully, such a reader has been disabused. The analysis we have provided illustrates several techniques central to the application of mathematics to the natural and the social sciences, as well as revealing some of the pitfalls.

We have shown that quite diverse problems may sometimes be reduced to the same mathematical formulation. In our case, we have regarded the problem variously as illustrating

Prediction: We think of 1, 2, 3 as describing the state of a system at successive times, for example the stock market index on successive days, and then our problem is to predict its next state.

Interpolation: We think of 1, 2, 3 as sampling a distribution, or as being experimental results in the investigation of the relationship between two measurable properties of, say, a physical system, and then our problem is to deduce the "true" distribution, or actual relation.

Optimisation: We think of 1, 2, 3 as being the result of a computational scheme, and we seeking the "most efficient" such scheme which produces this output — in our case we regarded a polynomial scheme as the most efficient, and for the polynomial ones, we regarded ones of lower degree more efficient than ones of higher degree.

We have shown that mathematics often provides more than one method to solve problems, by illustrating

continuous methods: when we sought to interpolate a polynomial function;
discrete methods: when we appealed to difference equations.

Moreover, we showed how particular parts of mathematical theory can be applied to facilitate computation. In our case, we saw how **linear algebra** can be applied.

Finally, our example of the Lotto draw hinted at the fact that **statistical theory** is often indispensable to the understanding of common phenomena.

Our sample problem offers natural paths to mathematical ideas using "real life" applications and shows that the abstractness of mathematics is the reason for, and not an obstacle to, its broad applicability.

This is the mathematics of the title.

As to the pitfalls, we have seen that the methods for solving problems often depend less on the mathematical expression of the problem than on additional requirements imposed by the nature of what is being modelled. Relying on the first formula to spring to mind for "the solution" is fraught with danger.

This should be a salutary warning to those relying on such instruments as "intelligence" tests, parts of which may be based on patent falsehoods. Such instruments are frequently used to direct the course of a child's formal education, not to mention their use in formulating policy and as the basis for expert evidence in courts of law. These facts make it imperative that such tests be soundly based on solid foundations, rather than crass misunderstanding of elementary mathematics.

This is the "mathematics" of the title.

School of Mathematics, Statistics and Computer Science, University of New England, Armidale NSW 2351
E-mail: imi@turing.une.edu.au

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