Book reviews

Ramanujan's Lost Notebook Part I

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1 Introduction

The meteor that was Ramanujan streaked across our skies from 1887 to 1920. He was a self–taught mathematical genius. During the time till he was about 25, he worked away at mathematics in his native India, recording his results in what has come down to us as three notebooks. More about the Notebooks later. He brought his work to the notice of G. H. Hardy, who with J. E. Littlewood recognised his genius, and eventually managed to persuade Ramanujan to come to Cambridge. Ramanujan spent five years in England working by himself and with Hardy, and published 21 papers, five jointly with Hardy. These have come down to us in the Collected Papers. During his last two years in England, Ramanujan became increasingly ill with what was treated as tuberculosis (despite there being indications to the contrary), and which now appears to have been (eminently curable!) amoebic hepatiasis. After two years rarely out of sanatoria, and after a slight apparent recovery, Ramanujan FRS returned to India, where, as far as I can tell, he was largely ignored; his illness grew worse, and after about a year, he died at the age of 32. During his last year, he worked at mathematics constantly, we are told. From that last year, all we have of his work is a sheaf of 138 pages (surely this must constitute only a fraction of his year's output?), which eventually came to reside in the library of Trinity College, Cambridge. Here it was discovered in the (northern) spring of 1976 by George E. Andrews. Andrews realised immediately its importance, and promptly named it the "Lost Notebook".

By now, quite a number of books and many papers have been written about Ramanujan and his work, but at the moment I would like to describe what happened with regard to his Notebooks and the Lost Notebook.

2 The Notebooks

It was proposed that Ramanujan's Notebooks (from his early years) be edited and published. The task was taken up by G. N. Watson and B. M. Wilson, but following Wilson's premature death, and with Watson losing interest after about 10 years, the project lapsed (though the work done by Watson and Wilson survived). In 1957 the Tata Institute of Fundamental Research put out a photocopy edition of the three Notebooks in two volumes. This is now completely unobtainable, but I was lucky and got a copy in the early 80's from a Bombay remainder(!) bookseller. Through an unlikely sequence of events, Bruce C. Berndt took up in 1977 where Watson and Wilson had left off, and over the next 30 years or so, published in five volumes an edited version of Ramanujan's Notebooks, in which he proved or gathered from the literature proofs of all 3254 claims made by Ramanujan.

3 The Lost Notebook

My first contact with the Lost Notebook came about when, in late 1978, Andrews came to Australia on Sabbatical. He showed me one result from the Lost Notebook, which I shall discuss below; the study of that one result earnt me about 40% of my PhD under Andrews, and I shall always be indebted to Andrews and Ramanujan for that!

At that time, if my memory serves me, Andrews had proved about 350 of the nearly 650 claims in the Lost Notebook.

In 1987, for the Centenary of the birth of Ramanujan, it was decided to put out a photocopy edition of the Lost Notebook, and whatever unpublished manuscripts, letters by and about Ramanujan the editors could lay their hands on. This was thrown together in a short time, and Andrews hastily wrote a masterful preface. Unfortunately, like the earlier Notebooks, the "Lost Notebook and Other Unpublished Papers" is utterly unobtainable.

At some stage, Andrews and Berndt decided to do for the Lost Notebook and Other Unpublished Papers (LNBaOUP) what Berndt had done for the Notebooks. This project is expected to occupy four volumes. So here, thirty years after the Lost Notebook was discovered, we have the first volume, chiefly on work done by Ramanujan in the one year in which he lay mortally ill!

4 The book under review

Whereas Ramanujan treated his Notebooks as an essentially systematic compendium of the results he found, and numbered them accordingly, with divisions into chapters, the Lost Notebook is a jumble, presumably written in something like chronological order of discovery. (Part of the problem is, I believe, caused by the fact that Ramanujan had all his ideas bubbling along simultaneously.) In particular, the entries are not numbered. It is not clear whether there is a canonical order to the sheets as determined by the librarian at Trinity College, or whether they are in any semblance of order. But by the time the sheets came to be photocopied, the editors took it upon themselves to try and improve matters by inserting among the pages of the Lost Notebook letters, fragments of previously unpublished and published manuscripts and manuscript copies (by others) of other manuscripts of Ramanujan that appeared to be related to work occurring nearby. The result is that the LNBaOUP is even more of a jumble. The one mitigating factor is that pages of the Lost Notebook are identified as such.

Now what Andrews and Berndt are doing is taking up the challenging task of sieving and sorting the many entries into chapters, and numbering the entries. They then proceed to give a proof of each entry, either by referring to work in the literature or by providing a new proof.

AB1 consists of 18 chapters, in which 442 enties are discussed. The first half of the book is essentially devoted to various aspects of continued fractions. Then we have three chapters on q–series, including the Rogers–Ramanujan identities and similar, then chapters on a variety of topics, including partial fractions, Hadamard products of q–series, integrals of theta functions, incomplete elliptic integrals, infinite integrals of q–products, modular equations and Lambert series. These are followed by a location guide, listing page by page of LNBaOUP all the entries on that page discussed in AB1, then a Provenance, listing the sources in the literature from which proofs have been taken for AB1 (in the main, papers by Andrews, Berndt and Berndt's students and colleagues) and a bibliography of 302 items (including the items in the Provenance).

Let me attempt to whet your appetite for AB1 by discussing a few of my points of contact with LNBaOUP. The mathematics may at first sight be daunting, but bear with it!

The claim of Ramanujan that I mentioned above is the following AB1 Entry 6.2.1,
If

$$G(a,b,\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(a+\lambda)\cdots(a+\lambda q^{n-1})q^{(n^2+n)/2}}{(1-q)\cdots(1-q^n)(1+bq)\cdots(1+bq^n)}$$

then

$$\frac{G(aq,b,\lambda q)}{G(a,b,\lambda)} = \frac{1}{1+}\,\frac{aq+\lambda q}{1+}\,\frac{bq+\lambda q^2}{1+}\,\frac{aq^2+\lambda q^3}{1+}\,\frac{bq^2+\lambda q^4}{1+\cdots}.$$

Andrews was the first to prove this, though "with some difficulty". When he came to Sydney he issued me the challenge of finding the convergents to the continued fraction, and then disappeared to Armidale for a week or two. He returned to find a note on his office door "I have found the formula, but this note is too small to contain it." This result appears as [6, Theorem 3].

If we let $n \to \infty$ in my result, we find

$$1 + \frac{aq+\lambda q}{1+}\,\frac{bq+\lambda q^2}{1+}\,\frac{aq^2+\lambda q^3}{1+}\,\frac{bq^2+\lambda q^4}{1+\cdots} = \frac{P(a,b,\lambda,q)}{P(b,aq,\lambda q,q)}$$

where

$$P(a,b,\lambda,q) = \sum_{r,s,t=0}^{\infty} \frac{a^r b^s \lambda^t q^{(r^2+r+s^2+s)/2+rs+rt+st+t^2}}{(q)_r(q)_s(q)_t}. \tag{1}$$

(Here $(q)_n = (1-q)\cdots(1-q^n)$ if $n \geq 1$, $(q)_0 = 1$.)

Now, $P(a,b,\lambda,q)$ is clearly symmetric in a,b, and it is easy to show that

$$P(a,b,\lambda,q) = \prod_{n=1}^{\infty}(1+bq^n)\cdot G(a,b,\lambda).$$

It follows that

$$1 + \frac{aq+\lambda q}{1+}\,\frac{bq+\lambda q^2}{1+}\,\frac{aq^2+\lambda q^3}{1+}\,\frac{bq^2+\lambda q^4}{1+\cdots} = \frac{P(a,b,\lambda,q)}{P(b,aq,\lambda q,q)} = \frac{P(a,b,\lambda,q)}{P(aq,b,\lambda q,q)}$$
$$= \frac{\prod_{n=1}^{\infty}(1+bq^n)\cdot G(a,b,\lambda)}{\prod_{n=1}^{\infty}(1+bq^n)\cdot G(aq,b,\lambda q)},$$

and we are done. (And we see that in some sense the result is about $P(a,b,\lambda,q)$ rather than $G(a,b,\lambda)$. Stating the result in terms of $G(a,b,\lambda)$ is a typical Ramanujan–esque twist, which, deliberately or not, made the result harder to prove.)

The authors give two later proofs, both starting with $P(a,b,\lambda,q)$. Both proofs rely on properties of $P(a,b,\lambda,q)$ that follow easily from (1). In my opinion, this is one instance where too much detail obscures the beauty and simplicity of Ramanujan's result (see my closing remarks).

The authors then proceed to state and prove one by one about 10 corollaries of Entry 6.2.1 given by Ramanujan. A number of these are special cases of just one result in the existing literature [6, Theorem 2], which the authors might have presented.

The celebrated Rogers–Ramanujan continued fraction is essentially

$$C(q) = 1 + \frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\cdots} = \frac{1+\sum_{n=1}^{\infty}(-1)^n(q^{(5n^2-n)/2}+q^{(5n^2+n)/2})}{1+\sum_{n=1}^{\infty}(-1)^n(q^{(5n^2-3n)/2}+q^{(5n^2+3n)/2})} \tag{2}$$
$$= \left(\begin{matrix} q^2, q^3 \\ q, q^4 \end{matrix}; q^5\right)_\infty.$$

Here we adopt the notation

$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$
$$(a_1;a_2,\cdots,a_k;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n,$$
$$\left(\begin{matrix} a_1,a_2,\ldots,a_k \\ b_1,b_2,\ldots,b_l \end{matrix};q\right)_n = \frac{(a_1,a_2,\cdots,a_k;q)_n}{(b_1,b_2,\cdots,b_l;q)_n}.$$

If we write

$$C(q) = \sum_{n=0}^{\infty} v_n q^n$$

then

$$\left(1+\sum_{n=1}^{\infty}(-1)^n(q^{(5n^2-3n)/2}+q^{(5n^2+3n)/2})\right)\sum_{n=0}^{\infty} v_n q^n$$
$$= 1+\sum_{n=1}^{\infty}(-1)^n(q^{(5n^2-n)/2}+q^{(5n^2+n)/2})$$

and it is straightforward to calculate the v_n up to some point. The same remark applies to

$$C(q)^{-1} = \sum_{n=0}^{\infty} u_n q^n.$$

In 1968 or thereabouts, George Szekeres carried out these calculations and observed that from some point on, the sign (+ or −) of the u_n and of the v_n are both periodic with period 5. He and Bruce Richmond succeeded in proving remarkable asymptotic formulae for the u_n and v_n,

$$u_n = \frac{\sqrt{2}}{(5n)^{3/4}}\exp\Big(\frac{4\pi}{5\sqrt{5n}}\Big)\left\{\cos\left(\frac{4\pi n}{5}+\frac{3\pi}{25}\right)+\mathrm{O}(n^{-1/2})\right\},$$
$$v_n = \frac{\sqrt{2}}{(5n)^{3/4}}\exp\Big(\frac{4\pi}{5\sqrt{5n}}\Big)\left\{\cos\left(\frac{2\pi n}{5}-\frac{4\pi}{25}\right)+\mathrm{O}(n^{-1/2})\right\}$$

which confirm Szekeres's observations.

There are some formulae in LNBaOUP, AB1 Entries 4.2.2 and 4.2.4 which Andrews proved and used to prove precise versions of Szekeres's observations, given as AB1 Corollaries 4.2.1 and 4.2.4. In order to prove Entries 4.2.2 and 4.2.4, one need only apply the quintuple product identity to the right hand sides of (2) and its reciprocal.

Remarkably, Ramanujan failed to notice that he could apply the quintuple product identity to Entries 4.2.2 and 4.2.4 and obtain simpler formulae. Indeed, it can be shown that [8]

$$\sum_{n\geq 0} v_{5n}q^n = 1/\prod_{n=1}^{\infty}(1-q^n)^{a_n}$$

where $a_n = 0$ if $n \equiv 0, \pm 7 \pmod{25}$, $a_n = 2$ if $n \equiv \pm 4 \pmod{25}$ and $a_n = 1$ otherwise, with similar formulae for all other nine sequences. From these, Corollaries 4.2.1 and 4.2.2 can be deduced directly.

Finally, it is worth noting that Ramanujan carried the calculation of the u_n and the v_n (correctly) just beyond 1000 and the results are tabulated on p.49 of LNBaOUP. My bet is that Ramanujan also observed the periodicity of sign, and that was his reason for carrying the calculation so far.

My final mathematical comments concern Chapter 10. Here we find a discussion of a fragment concerning the so–called Rogers–Ramanujan identities.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}.$$

Ramanujan discovered these before coming to England, but could not give a proof. In about 1916, he discovered proofs buried within an old paper by Rogers. Ramanujan simplified Rogers's proof, as did Rogers, and the two proofs were published together. Incidentally, Andrews once opined to me that the identities should be named after the three people who independently discovered and proved them, the Rogers–Schur–Baxter identities. The outward appearance of the fragment under consideration is that it was written to justify Ramanujan's belief that the identities held, but I get the feeling it was written *post facto*. In any case, Ramanujan gave four reasons, the fourth being incomplete, and the authors do their best to examine the fragment. In examining the first reason, rather than simply explaining Ramanujan's calculations, they reproduce in over four pages a proof of the R–R identities appealing to Bailey chains and a transformation due to Sears. Here, I think, a reference to Andrews's paper would have sufficed.

Incidentally, the first reason Ramanujan advances is that P. A. MacMahon had verified the identities up to q^{55}. Nowadays, using MAPLE 9.5 and Frank Garvan's "q–series" package, verifying the R–R identities to q^{1000} takes about one minute. (What could Ramanujan have accomplished with such tools?)

I might have commented on more of the book if I had understood more. I guess I will gradually pick up an understanding of more of Ramanujan's work by reading this book as well as Berndt's earlier books and the projected three additional books in this series, but always with pen and paper handy! There is a lot of work involved. I once read of Ramanujan's work that there is nothing that could be asked in even the most difficult examination in the world!

Just before closing, I will give you just two more intriguing formulae from LNBaOUP. If $R(q) = q^{\frac{1}{5}} C(q)^{-1}$ then

$$R(q) = \frac{\sqrt{5}-1}{2} \exp\left(-\frac{1}{5}\int_q^1 \frac{(1-t)^5(1-t^2)^5\cdots}{(1-t^5)(1-t^{10})\cdots}\frac{\mathrm{d}t}{t}\right)$$

and

$$R(q) = \frac{\sqrt{5}-1}{2} - \frac{\sqrt{5}}{1+\dfrac{3+\sqrt{5}}{2}\exp\left(\dfrac{1}{\sqrt{5}}\displaystyle\int_0^q \frac{(1-t)^5(1-t^2)^5\cdots}{(1-t^{1/5})(1-t^{2/5})\cdots}\frac{\mathrm{d}t}{t^{4/5}}\right)}.$$

I do have a couple of criticisms. The first is that with the authors' approach (that is, relying on the somewhat narrow base of their Provenance) they quite often miss the opportunity of giving the optimal presentation. The second is that the authors do not help us sufficiently to see what is easy, and what is difficult. Ramanujan wrote at many levels, and we need to know whether what we are studying is either fairly superficial, or deep. Thus they give too much detail in some proofs (thereby obscuring their simplicity and beauty), and give too little detail in others.

But these are perhaps minor points. Andrews and Berndt are to be congratulated on the valuable job they are doing. This is a first step (they would agree with me) on the way to an understanding of the work of the genius Ramanujan. It should act as an inspiration to future generations of mathematicians to tackle a job that will never be complete.

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