

Ramanujan's "most beautiful identity"

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You may have heard of Ramanujan (it is hard to believe you have not), but you may know little or nothing of his work. Of all the many identities he presented, Hardy chose one which for him represented the best of Ramanujan. I would like to show you this identity, and prove it. Following Euler, we define a partition of the positive integer n as a representation of n as a sum of positive integers, in which order is unimportant. The partitions of 4 are $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$. The number of partitions of n is denoted by $p(n)$; thus, $p(4) = 5$. For convenience, we define $p(0) = 1$. Euler showed that the partition generating function

$$P(q) = \sum_{n \geq 0} p(n)q^n = 1 + 1q + 2q^2 + 3q^3 + 5q^4 + \dots = \frac{1}{(q)_\infty},$$

where

$$(q)_\infty = \prod_{n \geq 1} (1 - q^n).$$

He also showed that

$$(q)_\infty = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + + - - \dots$$

In the series on the right, the terms occur in pairs, alternately with coefficients -1 and $+1$. The powers, 1, 2, 5, 7, 12, 15, ... are known as the pentagonal numbers, and Euler's expansion above as the pentagonal number theorem. The easiest way of generating the pentagonal numbers is to proceed as follows. The triangular numbers 1, 1 + 2, 1 + 2 + 3 and so on are

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, \dots$$

Some are divisible by 3, some not. If we divide by 3 those that are divisible by 3, we obtain the pentagonal numbers! Euler's pentagonal number theorem is the special case $a = 1$ of Jacobi's triple product identity when written

$$\begin{aligned} & (1 - a^{-1}q)(1 - aq^2)(1 - q^3)(1 - a^{-1}q^4)(1 - aq^5)(1 - q^6) \dots \\ & = 1 - a^{-1}q - aq^2 + a^{-2}q^5 + a^2q^7 - a^{-3}q^{12} - a^3q^{15} + + - - \dots \end{aligned}$$

(The triple product identity can be derived as a limiting special case of a two-variable identity of Euler. Also, it is important that I mention Ramanujan found a marvellous and powerful extension of the triple product identity.) Back to the main story: We have that the partition generating function is the reciprocal of the series $1 - q - q^2 + q^5 + q^7 - - + + \dots$

This implies

$$\begin{aligned} p(0) &= 1, \\ p(1) - p(0) &= 0, \\ p(2) - p(1) - p(0) &= 0, \\ p(3) - p(2) - p(1) &= 0, \end{aligned}$$

and, more generally, for $n > 0$,

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \dots = 0.$$

Here, you recognise the pattern of $+$ and $-$ signs, and the numbers 1, 2, 5, 7, \dots . For each n , the sum on the left terminates, since all terms with negative argument are zero. P. MacMahon, who was in Cambridge with Hardy and Ramanujan, used the above recurrence to calculate $p(n)$ for $n \leq 200$, and fortuitously listed the values in groups of five thus:

1	7	42	176	627	1958	·
1	11	56	231	792	2436	·
2	15	77	297	1002	3010	·
3	22	101	385	1255	3718	·
5	30	135	490	1575	4565	·

Ramanujan observed that the numbers at the bottom of each group are divisible by 5, that is, $5|p(5n+4)$. He also observed that $7|p(7n+5)$, $11|p(11n+6)$, and, on the basis of the very small amount of evidence provided by MacMahon's table, formulated a very general conjecture, which was essentially correct; the proof was completed by Oliver Atkin in the 1950's. Ramanujan did much more than prove $5|p(5n+4)$. We can write down the generating function of the $p(5n+4)$,

$$\sum_{n \geq 0} p(5n+4)q^n = 5 + 30q + 135q^2 + 490q^3 + 1575q^4 + 4565q^5 + \dots$$

Ramanujan's "most beautiful identity" says that this series can be written as a neat product,

$$\sum_{n \geq 0} p(5n+4)q^n = 5 \frac{(q^5)_\infty^5}{(q)_\infty^6}.$$

In short, if $(q)_\infty^{-1} = \sum p(n)q^n$ then $\sum p(5n+4)q^n = 5(q^5)_\infty^5 / (q)_\infty^6$. Note that this formulation does not require that we have a combinatorial interpretation of the $p(n)$; the statement can be regarded as purely algebraic. My aim is to sketch a proof of this wonderful identity. We can write, with $\omega \neq 1$ a fifth root of unity,

$$\frac{1}{(q)_\infty} = \frac{(\omega q)_\infty (\omega^2 q)_\infty (\omega^3 q)_\infty (\omega^4 q)_\infty}{(q)_\infty (\omega q)_\infty (\omega^2 q)_\infty (\omega^3 q)_\infty (\omega^4 q)_\infty}.$$

The denominator is

$$\begin{aligned} & \prod_{n \geq 1} (1 - q^n)(1 - \omega^n q^n)(1 - \omega^{2n} q^{2n})(1 - \omega^{3n} q^{3n})(1 - \omega^{4n} q^{4n}) \\ &= \prod_{5|n} (1 - q^n)^5 \cdot \prod_{5 \nmid n} (1 - q^n)(1 - \omega q^n)(1 - \omega^2 q^n)(1 - \omega^3 q^n)(1 - \omega^4 q^n) \\ &= \prod_{n \geq 1} (1 - q^{5n})^5 \cdot \prod_{5 \nmid n} (1 - q^{5n}) \\ &= \prod_{n \geq 1} (1 - q^{5n})^5 \cdot \prod_{n \geq 1} (1 - q^{5n}) / \prod_{5|n} (1 - q^{5n}) \\ &= (q^5)_\infty^6 / (q^{25})_\infty, \end{aligned}$$

and we have

$$\frac{1}{(q)_\infty} = \frac{(\omega q)_\infty (\omega^2 q)_\infty (\omega^3 q)_\infty (\omega^4 q)_\infty}{(q^5)_\infty^6 / (q^{25})_\infty}.$$

Now we deal with the numerator. The pentagonal numbers are 0, 1 or 2 modulo 5, and we can write

$$\begin{aligned} (q)_\infty &= (1 + q^5 - q^{15} - q^{35} - q^{40} - q^{70} \dots) \\ &\quad - q(1 - q^{25} - q^{50} + q^{125} \dots) \\ &\quad - q^2(1 - q^5 + q^{10} - q^{20} - q^{55} + q^{75} \dots). \end{aligned}$$

It is easy to prove that the central term is $-q(q^{25})_\infty$, and we can write

$$(q)_\infty = (q^{25})_\infty (a - q - q^2 b),$$

with a and b series in q^5 . An identity of Jacobi states

$$(q)_\infty^3 = 1 - 3q + 5q^3 - 7q^6 + \dots$$

where the powers on the right are the triangular numbers. The left-hand side is

$$\begin{aligned} & (q^{25})_\infty^3 (a - q - q^2 b)^3 \\ &= (a^3 - 3q^5 b^2) + q(-3a^2 - q^5 b^3) + q^2(3a - 3a^2 b) + q^3(6ab - 1) + q^4(3ab^2 - 3b). \end{aligned}$$

Since the triangular numbers are never 2 or 4 modulo 5, we have

$$q^2(3a - 3a^2 b) = 0 \quad \text{and} \quad q^4(3ab^2 - 3b) = 0,$$

and we obtain the remarkable fact that

$$ab = 1.$$

Ramanujan simply makes the claim (this is typical of him) that

$$a = \prod_{n \geq 0} \frac{(1 - q^{25n+10})(1 - q^{25n+15})}{(1 - q^{25n+5})(1 - q^{25n+20})} \quad \text{and} \quad b = \prod_{n \geq 0} \frac{(1 - q^{25n+5})(1 - q^{25n+20})}{(1 - q^{25n+10})(1 - q^{25n+15})}.$$

(It is not hard nowadays to give a proof of this, but Hardy regarded this as a gap in Ramanujan's proof, and said "Ramanujan never gave a complete proof.") We have

$$\begin{aligned}
 & (\omega q)_\infty (\omega^2 q)_\infty (\omega^3 q)_\infty (\omega^4 q)_\infty \\
 &= (q^{25})_\infty^4 (a - \omega q - \omega^2 q^2 b)(a - \omega^2 q - \omega^4 q^2 b) \\
 &\quad \times (a - \omega^3 q - \omega^6 q^2 b)(a - \omega^4 q - \omega^8 q^2 b) \\
 &= (q^{25})_\infty^4 \left((a^4 - q^5(2ab^2 + b)) + q(a^3 + q^5(ab^3 + b^2)) \right. \\
 &\quad \left. + q^2((a^3b + a^2) - q^5b^3) + q^3((2a^2b + a) + q^5b^4) \right. \\
 &\quad \left. + q^4(a^2b^2 + 3ab + 1) \right).
 \end{aligned}$$

So now we have

$$\begin{aligned}
 \sum_{n \geq 0} p(n)q^n &= \frac{(q^{25})_\infty^5}{(q^5)_\infty^6} \left((a^4 - q^5(2ab^2 + b)) + q(a^3 + q^5(ab^3 + b^2)) \right. \\
 &\quad \left. + q^2((a^3b + a^2) - q^5b^3) + q^3((2a^2b + a) + q^5b^4) \right. \\
 &\quad \left. + q^4(a^2b^2 + 3ab + 1) \right).
 \end{aligned}$$

If we extract those terms in which the powers are 4 modulo 5, we obtain

$$\sum_{n \geq 0} p(5n + 4)q^{5n+4} = \frac{(q^{25})_\infty^5}{(q^5)_\infty^6} \cdot q^4(a^2b^2 + 3ab + 1).$$

If we now use the fact that $ab = 1$, divide by q^4 and replace q^5 by q , we obtain

$$\sum_{n \geq 0} p(5n + 4)q^n = 5 \frac{(q^5)_\infty^5}{(q)_\infty^6}.$$

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